Date October 3, 2020

WEEK 1

Natural Numbers & Their Operation

\# Numbers keep a count of objects, i.e, 1, 2, 3, 4 ...

\# 0 to represent no objects at all.

\# Natural numbers : $\mathbb{N} = \{0, 1, 2, 3 \dots\}$

· sometimes $\mathbb{N}_0$ is used to emphasize 0 is included.

Operations –

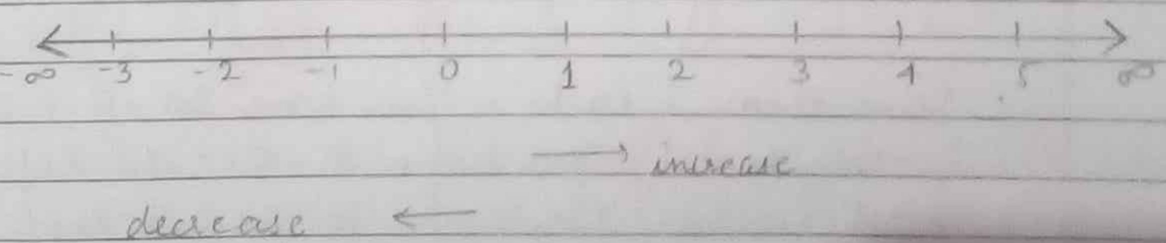
\# Addition, Subtraction, Multiplication, Division

· Which of these always produce a natural no. as the answer ?

Subtraction :

1. → 5 - 6 is not a natural no. (Subtraction fails)
→ Extend the natural no. with -ve no.s
→ -1, -2, -3 ....
→ INTEGERS : $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3 \dots\}$

Number Line :-



★ MULTIPLICATION & EXPONENTIATION

(2) → 7x4 - make 4 groups of 7

→ $m \times n = \underbrace{m + m + \dots + m}_{n \text{ times}}$

→ NOTATION : $m \times n$, $m \cdot n$, mn

→ Multiplication is repeated addition.

→ sign rule for multiplying negative no.s.

⇒ $-m \times n = -(m \cdot n)$
⇒ $-m \times -n = m \cdot n$

→ $m \times m = m^2$ - m squared
→ $m \times m \times m = m^3$ - m cubed
→ $m^k = \underbrace{m \times m \times m \times \dots m}_{k \text{ times}}$ - m to the power k

Multiplication is repeated addition.
Exponentiation is repeated multiplication.

★ DIVISION

(3) → You have 20 mangoes to distribute to 5 friends. How many do you give to each of them?

- Give them 1 each. You have 20 - 5 = 15 left
- Another round. You have 15 - 5 = 10 left
- Third round. You have 10 - 5 = 5 left
- Fourth round. You have 5 - 5 = 0 left
- 20 ÷ 5 = 4

→ Division is repeated subtraction.

→ What if you had only 19 mangoes to start with?

- After distributing 3 to each, you have 4 left.
- Can't distribute another round.
- The quotient of 19 ÷ 5 is 3
- The remainder of 19 ÷ 5 is 4
  ⇒ 19 mod 5 = 4

★ FACTORS

(4) → a divides b if b mod a is 0
- $a | b$
- b is a multiple of a   b is divisible by a

→ $4|20$, $7|63$, $32|1024$, ... $a|b$ → $4|16$
b mod a

→ $4 \nmid 19$, $9 \nmid 100$

→ a is a factor of b if $a|b$

→ Factors occur in pairs : factors of 12 are {1, 12}, {2, 6}, {3, 4}

→ ... unless the no. is a perfect square : factors of 36 are {1, 36}, {2, 18}, {3, 12}, {4, 9}, {6}.

★ PRIME NUMBERS

(5) → p is prime if it has only 2 factors $\{1, p\}$
· 1 is not a prime no. – only one factor.

→ Prime numbers are 2, 3, 5, 7, 11, 13 ......
· Sieve of Eratosthenes – remove multiples of p.

→ Every No. can be decomposed into prime factors.
→ $12 = 2 \times 2 \times 3 = 2^2 \times 3$
→ $126 = 2 \times 3 \times 3 \times 7 = 2 \times 3^2 \times 7$

→ This decomposition is unique – Prime Factorisation.

Summary

- **N** : natural numbers = $\{0, 1, 2, 3, \ldots\}$
- **Z** : integers = $\{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$
- Arithmetic Operations : $+, -, \times, \div, m^n$
- Quotient, remainder, a mod b
- Divisibility, $a|b$
- Factors
- Prime Numbers & Prime Factorisation

Date: October 4, 2020

RATIONAL NUMBERS

Division

→ Cannot represent $19 \div 5$ as integer

→ Fractions : $3\frac{4}{5}$

→ Rational Numbers : p/q, p and q are integers
· Numerator 'p', denominator 'q', $q \neq 0$
· Use Q to denote rational no.'s

→ The same no. can be written in many ways
· $3/5 = 6/10 = 30/50 \ldots$

→ Useful to add, subtract, compare rational no.'s
· $3/5 + 3/4 = 12/20 + 15/20 = 27/20$
· $3/5 < 3/4$ because $12/20 < 15/20$

→ Representation is not unique
· $3/5 = 6/10 = 30/50 \ldots$

→ Reduced form : $\frac{p}{q}$, where p, q have no common factors.
· Reduced form of $\frac{18}{60}$ is $\frac{3}{10}$

→ Greatest common divisor : gcd(18,60) = 6

- Recall prime factorisation
- 18 = 2·3·3 , 60 = 2·2·3·5
- Common prime factors are 2·3
- Can find gcd(m, n) more efficiently

Density

→ For each integer, we have a next integer and a previous integer

→ For m, next is m+1, previous is m-1

→ Next : No integer between m and m+1
Previous : No integer between m-1 and m.

→ Not possible for rationals
- Between any 2 rationals we can find another one.
- Suppose, $\frac{m}{n} < \frac{p}{q}$

Their average $\left(\frac{m}{n} + \frac{p}{q}\right)/2$ lies b/w them.

→ Rationals are dense, integers are discrete.

Summary

→ Q : rational no.s
→ p/q, where p, q are integers
→ Representation is not unique $p/q = n \cdot p / n \cdot q$
→ Reduced form, gcd(p,q) = 1
→ Rationals are dense - can't talk of next or prev.

Date October 4, 2020

REAL AND COMPLEX NO.s

Beyond Rationals

→ Rational numbers are dense
- B/w any 2 rationals we can find another one

→ Is every point on the number line a rational no?

→ For an integer m, its square is $m^2 = m \cdot m$

→ Square root of m, $\sqrt{m}$, is r such that $r \cdot r = m$

→ Perfect squares : 1, 4, 16, 9, 25, 36, 49, ... 256

→ Square roots : 1, 2, 4, 3, 5, 6, 7 16

→ What about integers that are not perfect squares?

→ $\sqrt{2}$ cannot be written as $\frac{p}{q}$

→ Yet, we can draw a line of length $\sqrt{2}$
- Diagonal of a square whose sides have length 1.

→ $\sqrt{2}$ is irrational

→ **Real Numbers : R** - all rational & irrational no.s

→ Like rationals, real no.s are dense
- If $r < r'$, then $\left(\frac{r+r'}{2}\right)$ lies b/w r & r'.

Beyond Reals

→ Some well known irrationals
- $\pi = 3.1415927\ldots$
- $e = 2.7182818\ldots$

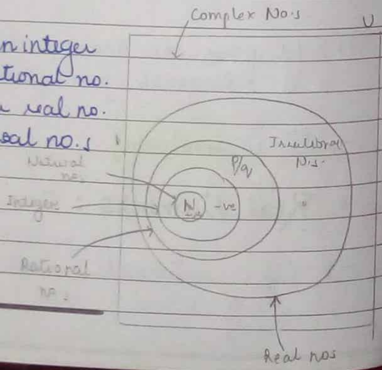
→ Can we stop with real no.s ?
- What about $\sqrt{-1}$
- For any real number r, r^2 must be +ve - (law of signs for multiplication)

→ $\sqrt{-1}$ is a complex number

Summary

→ Real no.s extend rational no.s
→ Typical irrational no.s - sq. root of integers that are not perfect sq.
→ Real numbers are dense like rationals.

- Every natural no. is an integer
- Every integer is a rational no.
- Every rational no. is a real no.
- Complex no. extend real no.s



October 5, 2020

SET THEORY

Sets

→ A set is a collection of items
- Days of the week : {Sun, Mon, Tues, Wed, Thurs, Fri, Sat}
- Factors of 24 : {1, 2, 3, 4, 6, 8, 12, 24}
- Prime no.s below 15 : {2, 3, 5, 7, 11, 13}

→ Sets may be infinite
- Different types of numbers : $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

→ No requirement that members of a set have uniform type.
- Sets of objects in a painting
- Sets of objects in a room

Order, Duplicates, Cardinality

→ Sets are unordered
- {Kohli, Dhoni, Pujara}
- {Pujara, Kohli, Dhoni}

→ Duplicates don't matter
- {Kohli, Dhoni, Pujara, Dhoni}

→ Cardinality : number of items in a set
- For finite sets, count the items
- {1, 2, 3, 4, 6, 8, 12, 24} has cardinality 8.

- What about infinite sets?
  - Is $\mathbb{Q}$ bigger than $\mathbb{Z}$?
  - Is $\mathbb{R}$ bigger than $\mathbb{Q}$?

Describing Sets, Membership

- Finite sets can be listed out explicitly
  - {Kohli, Dhoni, Pujara}
  - $\{1, 2, 3, 4, 6, 8, 12, 24\}$
- Infinite sets cannot be listed out
  - $\mathbb{N} = \{0, 1, 2, \ldots\}$ is not formal notation
- Not every collection of items is a set
  - Collection of all sets is not a set
  - Russell's Paradox
- Items in a set are called elements.
  - Membership : $x \in X$, x is an element of X
  - $5 \in \mathbb{Z}$, $\sqrt{2} \notin \mathbb{Q}$

Subsets

- X is a subset of Y if every element of X is also an element of Y
- NOTATION : $X \subseteq Y$
- Examples
  - {Kohli, Pujara} $\subseteq$ {Kohli, Pujara, Dhoni}
  - Primes $\subseteq \mathbb{N}$, $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Q} \subseteq \mathbb{R}$
- Every set is a subset of itself : $X \subseteq X$
  - $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$
- Proper Subset : $X \subseteq Y$ but $Y \neq X$
  - Notation : $X \subset Y$ or $X \subsetneq Y$
  - $\mathbb{N} \subsetneq \mathbb{Z}$, $\mathbb{Z} \subsetneq \mathbb{Q}$, $\mathbb{Q} \subsetneq \mathbb{R}$

The Empty Set & The Power set

- The empty set has no elements - ϕ
- $\phi \subseteq X$ for every set X
  - Every element of ϕ is also in X
- A set can contain other sets
- POWERSET - set of subsets of a set
  - $X = \{a, b\}$
  - Powerset is $\{\phi, \{a\}, \{b\}, \{a, b\}\}$
- Set with n elements has 2^n subsets
  - $X = \{x_1, x_2, x_3, \ldots x_n\}$
  - In a subset, either include or exclude each x_i
  - 2 choices per element, $\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n$ subsets

Subsets & Binary Numbers

→ $X = \{x_1, x_2, \ldots, x_n\}$

→ n bit binary numbers
  ⇢ 3 bits : 000, 001, 010, 011, 100, 110, 111

→ Digit i represents whether x_i is included in a subset
  ⇢ $X = \{a, b, c, d\}$
  ⇢ 0101 is $\{b, d\}$
  ⇢ 0000 is ϕ and 1111 is X

→ 2^n n bit binary numbers.

October 5, 2020

Construction Of Subsets And Set Operations

Constructing Subsets

★ Set Comprehension (Set Builder Form)

→ The subset of even integers
  $\{x \mid x \in \mathbb{Z}, x \bmod 2 = 0\}$

  ⇢ Begin with an existing set, $\mathbb{Z}$
  ⇢ Apply a condition to each element in that set
   . $x \in \mathbb{Z}$ such that $x \bmod 2 = 0$
  ⇢ Collect all the elements that match the condition

→ EXAMPLES
  ⇢ The set of perfect squares
   $\{m \mid m \in \mathbb{N}, \sqrt{m} \in \mathbb{N}\}$

  ⇢ The set of rationals in reduced form
   $\{p/q \mid p, q \in \mathbb{Z}, \gcd(p, q) = 1\}$

★ Intervals

→ Integers from -6 to +6 : $\{z \mid z \in \mathbb{Z}, -6 \le z \le 6\}$

→ Real number between 0 and 1

→ Closed Interval $[0,1]$ - include endpoints

→ $\{x \mid x \in \mathbb{R},\ 0 \le x \le 1\}$

→ Open Interval $(0,1)$ - exclude endpoints

→ $\{x \mid x \in \mathbb{R},\ 0 < x < 1\}$

→ Left Open $(0,1]$

→ $\{x \mid x \in \mathbb{R},\ 0 < x \le 1\}$

→ Right Open $[0,1)$

→ $\{x \mid x \in \mathbb{R},\ 0 \le x < 1\}$

Union, Intersection, Complement

→ Union - combine X and Y
$X \cup Y$
$\{a,b,c\} \cup \{c,d,e\} = \{a,b,c,d,e\}$

→ Intersection - elements common to X and Y
$X \cap Y$
$\{a,b,c,d\} \cap \{d,a,e,f\} = \{a,d\}$

→ Set Difference - elements in X that are not in Y,
$X - Y$ or $X \setminus Y$
$\{a,b,c,d\} - \{a,d,e,f\} = \{b,c\}$

→ Complement - elements not in X, $\bar{X}$ or X^c

→ Define complement relative to larger set, universe
→ Compliment of prime no.s in $\mathbb{N}$ are composite nos

Summary

→ Sets are a standard way to represent collections of mathematical objects.

→ Sets may be finite or infinite

→ Can carve out interesting subsets of sets

→ Set operations : union, intersection, difference, compliment.

October 6, 2020

SETS : Examples

Set Comprehension

→ square of the even integers $\{0, 4, 16, 36, 64, \dots\}$
$\{x^2 \mid x \in \mathbb{Z}, x \bmod 2 = 0\}$

- Generate Elements drawn from existing set
  -2, -1, 0, 1, 2, 3 ....
- Filter select elements that satisfy a constraint
  -2 0 2 4 6 ....
- Transform Modify selected elements
  4 0 4 16 36 ....

→ More Filters

- Rationals in reduced form
  $\{\frac{p}{q} \mid p/q \in \mathbb{Q}, \gcd(p,q) = 1\}$
- Reals in interval $[-1, 2)$
  $\{x \mid x \in \mathbb{R}, -1 \le x < 2\}$

→ cubes of first 5 natural numbers

$Y = \{n^3 \mid n \in \{0, 1, 2, 3, 4\}\}$

→ Cube of first 500 natural numbers ?

$Y = \{n^3 \mid n \in \{0, 1, 2, 3 \dots 498, 499\}\}$

→ Use set comprehension to define first 500 nat. nos

$X = \{n \mid n \in \mathbb{N}, n < 500\}$

→ Now a more readable version.

$X = \{n \mid n \in \mathbb{N}, n < 500\}$
$Y = \{n^3 \mid n \in X\}$

Perfect Squares

→ Integers whose square root is also an integer
$\{z \mid z \in \mathbb{Z}, \sqrt{z} \in \mathbb{Z}\}$

→ All squares are positive, so this is the same as
$\{n \mid n \in \mathbb{N}, \sqrt{n} \in \mathbb{N}\}$

→ Alternatively, generate all the perfect squares
$\{n^2 \mid n \in \mathbb{N}\}$

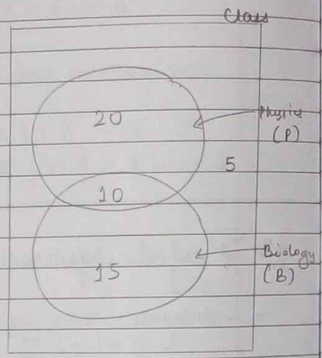
→ Extend the definition to rationals

- $9/16 = (3/4)^2$ is a square. $1/2 \neq (p/q)^2$ for any p, q
- $\{q \mid q \in \mathbb{Q}, \sqrt{q} \in \mathbb{Q}\}$ or $\{q^2 \mid q \in \mathbb{Q}\}$

Date: October 6, 2020

COUNTING PROBLEMS

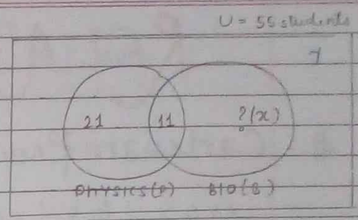
\# In a class, 30 students took Physics, 25 took Biology and 10 took both, 5 took neither. How many students are there in class?



- Draw sets for Physics (P) & Biology (B)
- 10 students are in $P \cap B$
- This leaves 20 students in $P \setminus B$. (Took Physics but didn't take Biology)
- Likewise 15 students in $B \setminus P$ (Took Biology but not Physics)
- 5 students in $\overline{P \cup B}$ (In the class but neither took Physics nor Biology)

→ Class Strength : $20+10+15+5 = 50$

\# In a class of 55 students, 32 students took Physics, 11 students took both Physics and Biology, and 7 took neither. How many students took Biology but not Physics?



Solution : $55 = x + 11 + 21 + 7$

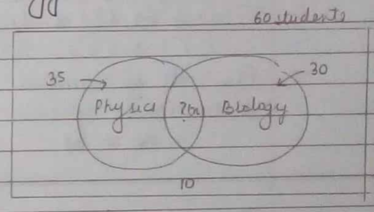
$55 = x + 39$

$x = 55 - 39$

$x = 16$

∴ 16 students took Biology but not Physics.

\# In a class of 60 students, 35 students took Physics, 30 took Biology, and 10 took neither. How many took both Physics and Biology?



- $P \cup B = n(P) + n(B) - n(P \cap B)$
- $P \cup B = 60 - 10 = 50$
- $n(P) = 35$

  $n(B) = 30$

  ∴ $n(P) + n(B) = 35 + 30 = 65$

Thus, $P \cap B = 65 - 50 = 15$

Summary :

∵ Set notation is useful way to concisely describe collection of objects

∵ Set comprehension combines generators, filters & transformations to produce new sets from old

∵ Venn diagrams can be useful to workout problems involving sets.

October 7, 2020

RELATIONS

Cartesian Product

→ $A \times B = \{(a,b) \mid a \in A, b \in B\}$

- Pair up elements from A and B
- $A = \{0,1\}$, $B = \{2,3\}$
- $A \times B = \{(0,2),(0,3),(1,2),(1,3)\}$

→ In a pair, the order is important
- $(0,1) \neq (1,0)$

→ For sets of numbers, visualize product as two dimensional space
- $\mathbb{N} \times \mathbb{N}$

Binary Relations

→ Select some pairs from the Cartesian product

→ Combine cartesian product with set comprehension

→ $\{(m,n) \mid (m,n) \in \mathbb{N} \times \mathbb{N}, n = m+1\}$
- $\{(0,1),(1,2),(2,3) \ldots (17,18) \ldots\}$

→ Pairs (d,n) where d is a factor of n
- $\{(d,n) \mid (d,n) \in \mathbb{N} \times \mathbb{N}, d \mid n\}$
- $\{(1,1), \ldots (2,82), \ldots (14,56), \ldots\}$

→ BINARY RELATION $R \subseteq A \times B$

→ Notation : $(a,b) \in R$, aRb

More Relations

→ Points at a distance 5 from $(0,0)$
- Distance from $(0,0)$ to (a,b) is $\sqrt{a^2+b^2}$
- $\{(a,b) \mid (a,b) \in \mathbb{R} \times \mathbb{R}, \sqrt{a^2+b^2} = 5\}$
- $\{(0,5),(5,0),(3,4),(-3,-4), \ldots\}$
- A circle with centre at $(0,0)$

→ Rationals in reduced form
- A subset of $\mathbb{Q}$
  - $\{\frac{p}{q} \mid (p,q) \in \mathbb{Z} \times \mathbb{Z}, \gcd(p,q) = 1\}$
- ... but also a relation on $\mathbb{Z} \times \mathbb{Z}$
  - $\{(p,q) \mid (p,q) \in \mathbb{Z} \times \mathbb{Z}, \gcd(p,q) = 1\}$

Types of Binary Relations

→ Identity Relation $I \subseteq A \times A$
- $I = \{(a,b) \mid (a,b) \in A \times A, a = b\}$

- $I = \{(a,a) \mid (a,a) \in A \times A\}$
- $I = \{(a,a) \mid a \in A\}$

→ Reflexive Relations

- $R \subseteq A \times A$, $I \subseteq R$
- $\{(a,b) \mid (a,b) \in \mathbb{N} \times \mathbb{N},\ a,b > 0,\ a \mid b\}$
  - $a \mid a$ for all $a > 0$

→ Symmetric Relations

- $(a,b) \in R$ if and only if $(b,a) \in R$
- $\{(a,b) \mid (a,b) \in \mathbb{N} \times \mathbb{N},\ \gcd(a,b) = 1\}$
- $\{(a,b) \mid (a,b) \in \mathbb{N} \times \mathbb{N},\ |a-b| = 2\}$

→ Transitive Relations

- If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$
- $\{(a,b) \mid (a,b) \in \mathbb{N} \times \mathbb{N},\ a \mid b\}$
  - If $a \mid b$ and $b \mid c$ then $a \mid c$
- $\{(a,b) \mid (a,b) \in \mathbb{R} \times \mathbb{R},\ a < b\}$
  - If $a < b$ & $b < c$ then $a < c$

→ Antisymmetric Relations

- If $(a,b) \in R$ and $a \neq b$, then $(b,a) \notin R$
- $\{(a,b) \mid (a,b) \in \mathbb{R} \times \mathbb{R},\ a < b\}$
  - If $a < b$ then $b \nless a$
- $M \subseteq P \times P$ relates mothers to children
  - If $(p,c) \in M$ then $(c,p) \notin M$

Equivalence Relations

→ Reflexive, symmetric & transitive

→ Same remainder modulo 5

- $7 \bmod 5 = 2$, $22 \bmod 5 = 2$
- If $a \bmod 5 = b \bmod 5$ then $(b-a)$ is a multiple of 5
- $\mathbb{Z} \bmod 5 = \{(a,b) \mid a,b \in \mathbb{Z},\ (b-a) \bmod 5 = 0\}$
- Divides integers into 5 groups based on remainder when divided by 5

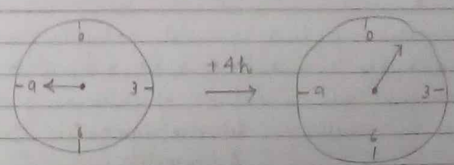
→ An equivalence relation partitions a set

→ Group of equivalent elements are called Equivalence Classes.

★ Measuring Time

clock displays hours modulo 12

2:00 am is equivalent to 2:00 pm



Beyond Binary Relations

→ Cartesian products of more than two sets

→ Pythagorean triplets

- Square on the hypotenuse is the sum of the squares on the opp. sides.
- $\{(a,b,c) \mid (a,b,c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N},\ a,b,c > 0,\ a^2+b^2=c^2\}$

→ Corners of squares

- A corner is a point $(x,y) \in \mathbb{R} \times \mathbb{R}$
- $((x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4))$ are related if they are four corners of a square.
- For instance:
  - $((0,0),(0,2),(2,2),(2,0))$
  - $((0.5,0),(0,0.5),(0.5,1),(1,0.5))$
- $Sq \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$

SUMMARY: → Cartesian products generate n-tupples from n-sets
- $(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times X_3 \ldots \times X_n$

→ A relation picks out a subset of Cartesian product

→ Properties of relations: Reflextive, Symmetric, Transitive, Asymmetric.

→ Equivalence relation partitions a set.

October 10, 2020

FUNCTIONS

→ A rule to map inputs to outputs

→ Convert x to x^2
- The rule $x \longrightarrow x^2$
- Give it a name: $sq(x) = x^2$
  - Input is a parameter

→ Need to specify the input and output sets

- Domain: Input set
  - domain$(sq) = \mathbb{R}$
- Codomain: Output set of possible values
  - codomain$(sq) = \mathbb{R}$
- Range: Actual values that the output can take
  - range$(sq) = \mathbb{R}_{\geq 0} = \{r \mid r \in \mathbb{R},\ r \geq 0\}$

→ $f: X \to Y$, domain of f is X, codomain is Y

Functions & Relations

→ Associate a relation R_f with each function f

→ $R_{sq} = \{(x,y) \mid x,y \in \mathbb{R},\ y = x^2\}$
- Additional notation: $y \stackrel{?}{=} x^2$

→ $R_f \subset \text{domain}(f) \times \text{range}(f)$

→ Properties of R_f

- defined on the entire domain
  - → for each $x \in$ domain(f), there is a pair $(x,y) \in R_f$
- Single Valued
  - → for each $x \in$ domain(f), there is exactly one $y \in$ codomain(f) such that $(x,y) \in R_f$

→ Drawing f as a graph is plotting R_f.

Lines

→ $f(x) = 3.5x + 5.7$

- → 3.5 is the slope
- → 5.7 is intercept where line crosses y-axis when $x=0$

→ Changing the slope and intercept produce different lines

- → $f(x) = 3.5x - 1.2$
- → $f(x) = 2x + 5.7$
- → $f(x) = -4.5x + 2.5$

→ In all these cases

- Domain = $\mathbb{R}$
- codomain = Range = $\mathbb{R}$

More Functions

→ $x \to \sqrt{x}$

→ Is this a function?

- → $5^2 = (-5)^2 = 25$
- → $\sqrt{25}$ = gives two options
- → By convention take the square root

→ What is the domain?

- → Depends on codomain
- → Negative no.s do not have real sq. roots
- → If codomain is $\mathbb{R}$, domain is $\mathbb{R} \geq 0$
- → If codomain is the set $\mathbb{C}$ of complex no.s, domain is $\mathbb{R}$.

Types Of Functions (on the basis of mapping)

→ Injective : Different inputs produces diff outputs (one-one)

- → If $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$
- → $f(x) = 3x + 5$ is injective
- → $f(x) = 7x^2$ is not ; for any a, $f(a) = f(-a)$

→ Surjective : Range is the codomain - (onto)

- → For every $y \in$ codomain(f), there is an $x \in$ domain(f) such that $f(x) = y$
- → $f(x) = -7x + 10$ is surjective
- → $f(x) = 5x^2 + 3$ is not surjective for codomain $\mathbb{R}$
- → $f(x) = 7\sqrt{x}$ is not surjective for codomain $\mathbb{R}$.

→ Bijective : 1-1 correspondance between domain and codomain.

- Every $x \in$ domain(f) maps to a distinct $y \in$ codomain
- Every $y \in$ codomain(f) has a unique pre-image $x \in$ domain(f) such that $y = f(x)$

THEOREM
A function is bijective if and only if it is injective and surjective.

→ From the definition, if a function is bijective it is injective and surjective.

→ Suppose a function f is injective and surjective

- Injectivity gurantees that f satisfies the first condition of bijection.
- Surjectivity says every $y \in$ codomain(f) has a pre image. Injectivity gurantees this pre-image is unique.

Bijections & Cardinality

→ For finite sets, we can count the items

→ What if we have two large sacks filled with marbles?

- Do we need to count the marbles in each sack?
- Pull out marbles in ~~each~~ ~~sack~~ pairs, one from each sack
- Do both sets become empty simultaneously?
- Bijection between the marbles in the sacks

→ For infinite sets

- No. of lines is the same as $\mathbb{R} \times \mathbb{R}$
- Every line $y = mx + c$ is determined uniquely by (m, c) and vice versa.

→ For every pair of points (x_1, y_1) and (x_2, y_2), there is a unique line passing through both points.

→ No. of lines is same as cardinality of $\mathbb{R} \times \mathbb{R}$

→ Does this show that $(\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}) = \mathbb{R}^2 \times \mathbb{R}^2$ has the same cardinality as $\mathbb{R} \times \mathbb{R}$?

→ The correspondance is not a bijection - many pairs of points describe the same line

Summary

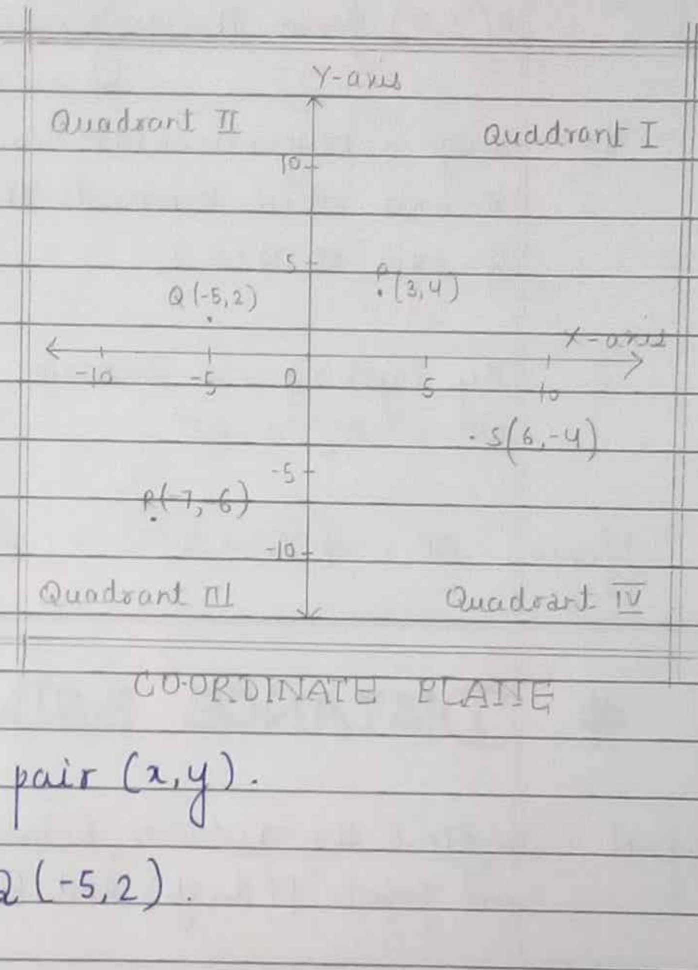
- A fn is given by a rule mapping inputs to outputs.
- Define the domain, codomain & range.
- Associate a relation R_f with each fn f.
- Properties of fn : Injective (one-one), Surjective (onto)
- Bijections : injective & surjective
- A bijection establishes that domain & codomain have same cardinality.

Date October 19, 2020

WEEK 2

Rectangular Co-ordinate System

- The horizontal line is called X-axis.
- The vertical line is called Y-axis
- The point of intersection of these two lines is called Origin.



COORDINATE PLANE

- Any point on the co-ordinate plane can be represented by an ordered pair (x, y).
- For example, $P(3,4)$, $Q(-5,2)$.
- The coordinate axes split the coordinate plane into four quadrants and two axes.

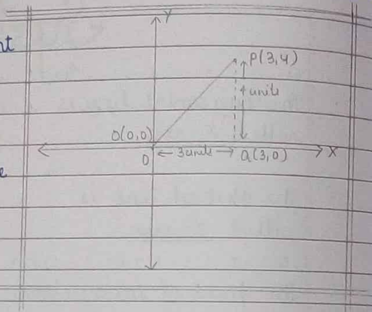
→ Quadrant I : $(+,+)$
→ Quadrant II : $(-,+)$
→ Quadrant III : $(-,-)$
→ Quadrant IV : $(+,-)$

→ X-axis : $(\pm, 0)$
→ Y-axis : $(0, \pm)$
→ Origin : $(0, 0)$

October 19, 2020

Distance Formula

DISTANCE OF A POINT FROM ORIGIN



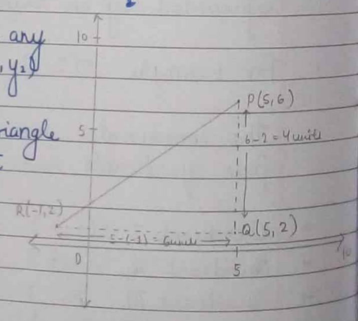
Goal: To find the distance of point P(3,4) from the origin.

1. Drop a perpendicular on X-axis which intersects the X-axis at Q(3,0).

2. By Pythagorean Theorem,
$OP^2 = OQ^2 + QP^2$

Hence, $OP = \sqrt{OQ^2 + QP^2} = \sqrt{3^2 + 4^2} = 5$

DISTANCE BETWEEN ANY 2 POINTS



Goal: To find the distance between any two points $P(x_2, y_2)$ and $R(x_1, y_1)$

1. Construct a right-angled triangle with right angle at point $Q(x_2, y_1)$

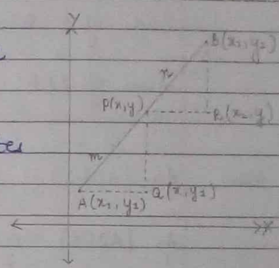
2. By Pythagoras Theorem,
$PR^2 = QR^2 + PQ^2$

$\Rightarrow PR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ ← Distance Formula

October 19, 2020

Section Formula

Given that, the point P cuts the line segment AB in the $m:n$ ratio. Our goal is to find the coordinates of P.



Let the co-ordinates of A and B are (x_1, y_1) and (x_2, y_2), respectively.

Assume that P has ~~no~~ the coordinates (x, y).

Observe the $\Delta AQP \sim \Delta PRB$,
Hence,

$$\frac{m}{n} = \frac{AP}{PB} = \frac{AQ}{PR} = \frac{PQ}{BR}$$

$$\Rightarrow \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

$$\therefore x = \frac{mx_2 + nx_1}{m+n}, \quad y = \frac{my_2 + ny_1}{m+n}$$

October 19, 2020

Area of a Triangle

Goal : To find the area of ΔABC with known coordinates.

Let the coordinates of the vertices be $A(x_1, y_1)$, $B(x_2, y_2)$ & $C(x_3, y_3)$.

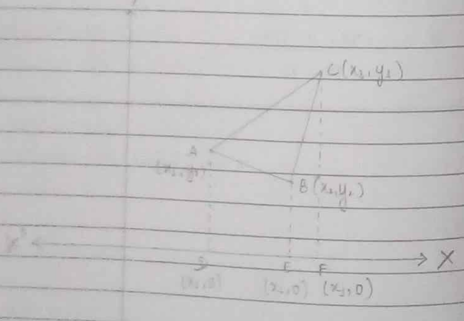
Area of ΔABC = Area of trap. ADFC − Ar. of trap ADEB − Ar. of trap. BEFC

Now, $Ar.(ADFC) = \frac{1}{2}(AD + CF) \times DF = \frac{1}{2}(y_1 + y_3)(x_3 - x_1)$

$Ar.(ADEB) = \frac{1}{2}(AD + EB) \times DE = \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$

$Ar.(BEFC) = \frac{1}{2}(BE + CF) \times EF = \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$

Thus, $Ar.(\Delta ABC) = \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$

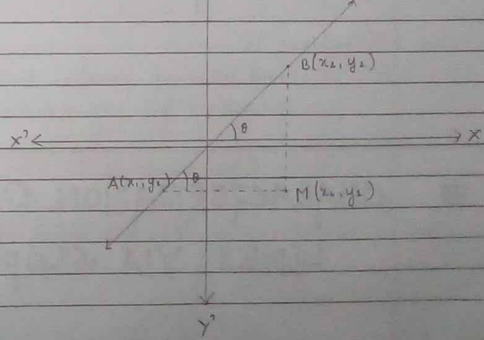


October 19, 2020

Slope of A Line

Goal : To find the slope of a line, given on a coordinate plane

- Identify two points on the line, say, $A(x_1, y_1)$ and $B(x_2, y_2)$.
- Construct a right angled triangle with a right angle at the point $M(x_2, y_1)$.
- Define
$$m = \frac{MB}{AM} = \frac{y_1 - y_2}{x_1 - x_2} = \tan\theta$$
- The 'm' is called slope of a line.
- θ is called the inclination of line with positive X-axis, measured in anti-clockwise direction.
- $0^\circ \le \theta \le 180^\circ$



- Observe that the lines parallel to X-axis have inclination of $0°$.
Hence the slope $m = \tan 0 = 0$

- The inclination of a vertical line is $90°$.
Hence, the slope m is undefined.

Definition : If θ is the inclination of a line l, then $\tan\theta$ is called the slope or gradient of line l.

If $\theta \neq 90°$, then $m = \tan\theta$

For obtuse,

$$m = \tan(180-\theta) = -\tan\theta = \frac{y_1 - y_2}{x_1 - x_2}$$

Can a slope of a line uniquely determine a line?
→ No, it can not uniquely determine the line.

How is the slope useful?

→ To explore :
- conditions of parallel lines
- conditions for perpendicular lines.

CHARACTERIZATION OF PARALLEL LINES VIA SLOPE

Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclination α and β resp.

- If l_1 is parallel to l_2, then $\alpha = \beta$
- It is clear that $\tan\alpha = \tan\beta$
- Hence, $m_1 = m_2$

- Assume, $m_1 = m_2$. Then $\tan\alpha = \tan\beta$
- Since $0° \leq \theta \leq 180°$, $\alpha = \beta$.
- Therefore, $l_1 \parallel l_2$.

Two non vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

CHARACTERIZATION OF PERPENDICULAR LINES VIA SLOPE

Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclinations α and β resp.

- If l_1 is perpendicular to l_2, then $90° + \alpha = \beta$
- Now, $\tan\beta = \tan(90° + \alpha) = -\cot\alpha = -1/\tan\alpha$
- Hence, $m_2 = -1/m_1 \Rightarrow m_1 m_2 = -1$

- Assume $m_1 m_2 = -1$. Then $\tan\alpha \tan\beta = -1$
- $\tan\alpha = -\cot\beta = \tan(90+\beta)$ or $\tan(90-\beta)$
- Hence, α and β differ by $90°$ which proves l_1 is perpendicular to l_2.

Two non-vertical lines l_1 and l_2 are perpendicular if and only if $m_1 m_2 = -1$.

ANGLES B/W TWO LINES

Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclinations α_1 and α_2 respectively.

Suppose l_1 and l_2 intersect and let ϕ and θ be the adjacent angles formed by l_1 and l_2.

Now, $\theta = \alpha_2 - \alpha_1$, for $\alpha_1, \alpha_2 \neq 90°$

Then, $\tan\theta = \tan(\alpha_2 - \alpha_1)$

$$\Rightarrow \tan\theta = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_2 \tan\alpha_1} = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Now, $\tan\phi = \tan(180 - \theta) = -\tan\theta = -\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$

COLLINEARITY OF POINTS

Consider 3 points A, B and C.

If slope of AB = slope of BC (common point B)

Thus, the three points A, B & C are collinear.

October 20, 2020

Representation of a Line

- How to represent a line uniquely?
- Given a point, how to decide whether the point lies on a line?

In other words, for a given line l, we should have a definite expression that describes the line in terms of coordinate plane.

If the coordinates of a given point P, satisfy the expression for the line l, then the point P lies on the line l.

HORIZONTAL & VERTICAL LINES

Horizontal Lines : A line is a horizontal line only if it is parallel to x-axis.

- To locate such a line, we need to specify the value it takes on Y-axis.
- That is, the expression for such a line is of the form $y = a$
- Then all points that lie on this line are of the form (x, a).

Vertical Lines : A line is a vertical line only if it is parallel to Y-axis.

- To locate such a line we need to specify the value it takes on X-axis.
- That is, the expression for such a line is of the form $x = b$.
- Then, all points that lie on this line are of the form (b, y).

EQUATION OF LINE

(1) Point slope Form

For a non-vertical line l, with slope m and a fixed point $P(x_0, y_0)$ on the line, can we find the equation (algebraic representation) of the line?

- Let $Q(x, y)$ be an arbitrary point on line l. Then, the slope of the line is given by

$$m = \frac{y - y_0}{x - x_0}$$

$$\Rightarrow \boxed{y - y_0 = m(x - x_0)}$$

Any point $P(x, y)$ is on line l, if and only if the coordinates of P satisfy the above equation.

(2) Two point Form

Let the line l pass through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Assume that $R(x, y)$ is an arbitrary point on the line l.

Then, the points P, Q & R are collinear.

Hence, slope of PR = slope of PQ

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$$

Any point $R(x, y)$ is on line l, if and only if, the coordinates of R satisfy the above equation.

(3) Slope - Intercept Form

$\Rightarrow$ Let a line l with slope m cut Y-axis at c. Then c is called y-intercept of the line l. That is, the point $(0, c)$ lies on the line l.

$\therefore$ By Point slope Form, $y - c = mx \Rightarrow \boxed{y = mx + c}$

$\Rightarrow$ Let a line l with slope m cut X-axis at d. Then d is called x-intercept of the line l. That is, the point $(d, 0)$ lies on the line l.

∴ By point slope form, $\boxed{y = m(x-d)}$

(4) Intercept Form

Suppose a line makes x-intercept at a and y-intercept at b. Then the two points on the line are (a,0) & ~~b,0~~ (0,b).

Using two point form,

$$y - 0 = \frac{b-0}{0-a}(x-a)$$

$$\Rightarrow \boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

Date: October 31, 2020

WEEK 3

General Equation of Line

	Different Forms Of Equation of Line	Representation
1.	Slope - Point Form	$m = \frac{y - y_0}{x - x_0}$
2.	Slope - Intercept Form	(a) $y = mx + c$ (b) $y = m(x-d)$
3.	Two Point Form	$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$
4.	Intercept Form	$\frac{x}{a} + \frac{y}{b} = 1$
5.	Normal Form	$x\cos\omega + y\sin\omega = p$

General Form Of Line Equation

$$\boxed{Ax + By + C = 0}$$

(1) Slope - point form

$$m = \frac{y - y_0}{x - x_0} \Rightarrow mx - mx_0 = y - y_0$$

classmate
Date
Page

$\Rightarrow mx - y - mx_0 + y_0 = 0$

Comparing with $Ax + By + C = 0$

$$\boxed{\begin{aligned} &\rightarrow m = \frac{-A}{B} \\ &\rightarrow y_0 - mx_0 = \frac{-C}{B} \end{aligned}}$$

(2) Slope Intercept Form

$y = mx + c$ $\qquad\qquad$ $y = m(x-d) \Rightarrow y = mx - md$

$\Rightarrow mx - y + c = 0$ $\qquad\qquad$ $\Rightarrow mx - y - md = 0$

comparing with $Ax + By + C = 0$,

$$\boxed{m = \frac{-A}{B} \;,\; c = \frac{-C}{B} \;\text{ or }\; d = \frac{-C}{A}}$$

(3) Two Point Form

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow (y-y_1)(x_2-x_1) = (x-x_1)(y_2-y_1)$$

$$\Rightarrow x_2 y - x_1 y - x_2 y_1 + \cancel{x_1 y_1} = x(y_2 - y_1) - x_1 y_2 + \cancel{x_1 y_1}$$

$$\Rightarrow y(x_2 - x_1) - x(y_2 - y_1) - x_2 y_1 + x_1 y_2 = 0$$

Comparing with $Ax + By + C = 0$

$$\boxed{\frac{y_2 - y_1}{x_2 - x_1} = \frac{-A}{B} \;,\; y_1 + \frac{A}{B}x_1 = \frac{-C}{B}}$$

classmate
Date
Page

4. Intercept Form

$$\frac{x}{a} + \frac{y}{b} = 1$$

$\Rightarrow bx + ay = ab$

$\Rightarrow bx + ay - ab = 0$

Comparing with $Ax + By + C = 0$

$$\boxed{a = \frac{-C}{A} \;,\; b = \frac{-C}{B}}$$

Conditions for A Parallel & Perpendicular Line

If the lines given are $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$

then, for the lines to be PARALLEL,

$$\boxed{a_1 b_2 = a_2 b_1}$$

and, for the lines to be PERPENDICULAR,

$$\boxed{a_1 a_2 + b_1 b_2 = 0}$$

November 2, 2020

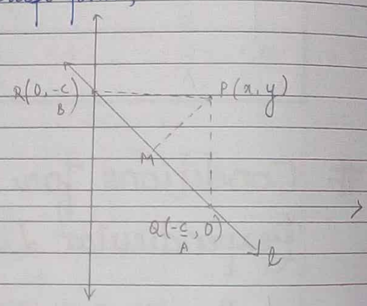
DISTANCE OF A POINT FROM A LINE

Goal: To find the distance of a point $P(x_1, y_1)$ from the line l having equation $Ax + By + C = 0$

$\therefore$ For $A, B \neq 0$, using intercept form,

x- intercept $= -\frac{C}{A}$

y- intercept $= -\frac{C}{B}$



$Ar.(\Delta PQR) = \frac{1}{2}(QR \times PM)$

$\Rightarrow PM = 2 \frac{Area(\Delta PQR)}{QR}$

$$A(\Delta PQR) = \frac{1}{2}\left|x_1\left(\frac{-C}{B}\right) - \frac{C}{A}\left(y_1 + \frac{C}{B}\right)\right| = \frac{1}{2}\frac{|C|}{|AB|}|Ax_1 + By_1 + C|$$

$$QR = \sqrt{\frac{C^2}{A^2} + \frac{C^2}{B^2}} = \frac{|C|}{|AB|}\sqrt{A^2+B^2}$$

Thus $$PM = \frac{2\,A(\Delta PQR)}{QR} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

DISTANCE B/W TWO PARALLEL LINES

Let l_1 and l_2 be two parallel lines with slopes m.

l_1 : $y = mx + c_1$. Comparing with general form, we get x-intercept at $\left(\frac{-c_1}{m}\right)$!

l_2 : $y = mx + c_2$. Comparing with general form, we get $A = -m$, $B = 1$ and $C = -c_2$

By using distance of a point from a line formula, where point is $\left(\frac{-c_1}{m}, 0\right)$, we get

$$\frac{|A(-c_1/m) + B(0) + C|}{\sqrt{A^2 + B^2}} = \frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$$

For general form, $m = \frac{-A}{B}$, $c_1 = -\frac{C_1}{B}$ and $c_2 = \frac{-C_2}{B}$

then

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$$

Distance of a Set of Points from a line

Apart from perpendicular distance, we can also talk about the distance which is parallel to Y-axis.

Consider the set of points $\{(x_i, y_i) \mid i = 1, 2, \ldots, n\}$ and a line with equation $y = mx + c$.

Then the squared sum of the distance of set of points from the line is defined as

$$SSE = \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

Least Squares Motivation

- In general, this raises the following question
- Given a set of points, how to find the line that fits the given set of points?
- In other words, what is the equation of the best fit line for given set of points?

In other words, if I need to find the equation of line $y = mx + c$, then the question can be reframed into two questions.

- What is the value of m & c that best fits the given set of points.
- What is the meaning of best fit?

BEST FIT : Given a set of n points, $\{(x_i, y_i) \mid i = 1, 2, \ldots n\}$ define

$$SSE = \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

Find the value of m & c that minimizes SSE.

Week 4

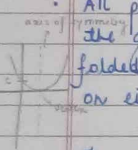
Quadratic Function

- A quadratic function is described by an equation of the form

$$f(x) = \underbrace{ax^2}_{\text{Quadratic Term}} + \underbrace{bx}_{\text{Linear Term}} + \underbrace{c}_{\text{Constant Term}}, \quad \text{where } a \neq 0$$

- The graph of any quadratic function is called **PARABOLA**.

IMPORTANT OBSERVATIONS



- All parabolas have an $\boxed{\text{axis of symmetry}}$. That is, if the graph paper containing the graph of parabola is folded along the axis of symmetry the portion of parabola on either sides will exactly match each other.

- The point at which the axis of symmetry intersects the parabola is called the vertex.

- The y-intercept of a quadratic function is c.

Let $f(x) = ax^2 + bx + c$, where $a \neq 0$

→ The y-intercept : $y = a(0)^2 + b(0) + c = c$

→ The eqⁿ of axis of symmetry : $x = -b/2a$

→ The x-coordinate of vertex : $\dfrac{-b}{2a}$

Maximum & Minimum Values

The y-coordinate of the vertex of a given quadratic function is the minimum or maximum value attained by the function.

The graph of a quadratic function $f(x) = ax^2 + bx + c$, where $a \neq 0$ is :

- Opens up and has a minimum value, if $a > 0$.
- Opens down and has maximum value, if $a < 0$

- The range of a quadratic function is

$$R \cap \{f(x) \mid f(x) \geq f_{min}\} \quad \text{or}$$

$$R \cap \{f(x) \mid f(x) \leq f_{max}\}$$

Slope of a Quadratic Function

Slope of $f = 2ax + b$

Slope denotes the rate of change of y with respect to x.

Hence, slope = 0 means the fn has either maximum or minimum which happens when

$$2ax + b = 0 \Rightarrow \boxed{x = \frac{-b}{2a}}$$

Roots of Equations & Zeroes of Functions

The solution to a quadratic equation are called roots of the equation.

One method for finding the roots of a quadratic eqn is to find zeroes of the related quadratic fn.

Observe that the zeroes of a function are x-intercepts of its graph and these are the solutions of related eqn. as $f(x) = 0$ at these points.

Date: January 6, 2021

WEEK 5

Quadratic Equations

Solutions of A Quadratic Equation Using Factorisation

★ Quadratic Function : Intercept Form

Let $y = f(x) = a(x-p)(x-q)$, where p and q represents x-intercepts for the function.
Then the form $y = a(x-p)(x-q)$ is called the INTERCEPT FORM.

★ Intercept form to the Standard form

Changing intercept form to standard form requires us to use FOIL method which can be described as follows:

The product of 2 binomials is the sum of the products of the first (F) terms, the outer (O), the inner (I) and the last terms.

$$(ax+b)(cx+d) = \underbrace{ax\cdot cx}_{F} + \underbrace{ax\cdot d}_{O} + \underbrace{cx\cdot b}_{I} + \underbrace{b\cdot d}_{L}$$

Quick Observations :
→ The product of coefficient of x^2 and the constant is abcd.
→ The product of two terms in the coefficient of x is also abcd.

Example : Write a quadratic eqⁿ with roots, $\frac{2}{3}$ and -4, in the standard form.

Recall: by standard form we mean $ax^2+bx+c=0$; $a,b,c \in \mathbb{Z}$

We have, by intercept form, $\left(x-\frac{2}{3}\right)(x+4) = 0$

$\therefore$ By Foil,
The standard form will be,

$$x^2 + 4x - \frac{2}{3}x - \frac{8}{3} = 0$$

$$\Rightarrow x^2 + x\left(4-\frac{2}{3}\right) - \frac{8}{3} = 0$$

$$\Rightarrow x^2 + \frac{10}{3}x - \frac{8}{3} = 0$$

$$\Rightarrow \boxed{3x^2 + 10x - 8 = 0}$$

★ Standard to Intercept Form

Example : Convert the fn $f(x) = 5x^2 - 13x + 6$ to intercept form.

Solution :
$$\begin{aligned} f(x) &= 5x^2 - 13x + 6 \quad \text{standard form} \\ &= 5x^2 - 10x - 3x + 6 \\ &= 5x(x-2) - 3(x-2) \\ &= (5x-3)(x-2) \\ &= 5\left(x-\frac{3}{5}\right)(x-2) \quad \text{intercept form} \end{aligned}$$

SOLUTIONS OF A QUADRATIC EQᴺ USING SQUARE

★ Solving a quadratic Equation by completing the Square method.

Example : $x^2 + 10x = 24$

Observe that $(x+a)^2 = x^2 + 2ax + a^2$.
Using this, write $10 = 2\times5$ and add 25 on both the sides of the equation to get,

$$\begin{aligned} x^2 + 25 + 10x &= 24 + 25 \\ (x+5)^2 &= 49 \\ \Rightarrow x+5 &= \pm\sqrt{49} \end{aligned}$$

Either $x+5 = -7$ or $x+5 = 7$
$\Rightarrow \boxed{x = -12}$ or $\boxed{x = 2}$

Quadratic Formula

$$ax^2 + bx + c = 0$$

$$\Rightarrow x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$\Rightarrow x^2 + \left(\frac{b}{a}\right)x = \frac{-c}{a}$$

$$\Rightarrow x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\Rightarrow x^2 + \left(\frac{b}{a}\right)x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\Rightarrow x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

discriminant

$$\boxed{D = \sqrt{b^2 - 4ac}}$$

$$\Rightarrow \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$ finding roots by Quadratic Formula

If $D > 0$, 2 roots
$D < 0$, no real root
$D = 0$, 1 root (2 equal roots)

Value of Discriminant	Type & no. of roots
$b^2 - 4ac > 0$ (perfect square)	→ 2 real, rational roots
$b^2 - 4ac > 0$ (not perfect square)	→ 2 real, irrational roots
$b^2 - 4ac = 0$	→ 1 real root
$b^2 - 4ac < 0$	→ No real roots

Summary of Concepts

METHOD	CAN BE USED	WHEN PREFERRED
Graphing	Occasionally	Best used to verify the answer found algebraically
Factoring	Occasionally	If constant term is 0 or factors are easy to find.
Completing the square	Always	Use when b is Even
Quadratic Formula	Always	Use when other methods fail.

Date January 14, 2021

WEEK 6

Polynomials

What is a Polynomial?

A Layman's Perspective : a polynomial is one kind of mathematical expression which is a sum of several mathematical terms.

→ Each term in this expression is called 'monomial' and the term can be a number, a variable or a product of several variables.

A Mathematician's Perspective : a polynomial is an algebraic expression in which the only arithmetic is addition, subtraction, multiplication and "natural" exponents of variables.

Examples : $4x^3 + 9x + 3 = 0$; $2x + 1 = 8$ ... etc.

Why do we call them Polynomials?

The word 'polynomial' is derived from two words
Poly + Nomen
many name / term

- Each term is called monomial
- A polynomial having 2 terms is called binomial.
- A polynomial with three terms is called trinomial.

E.g: A polynomial in 1 variable can be treated as,

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = \sum_{m=0}^{n} a_m x^m$$

exponent
variable
coeff. of term

Identification of Polynomials

→ Identify whether the following are polynomials or not.

1. $x^2 + 4x + 2$ → Yes
2. $x + x^{1/2}$ → No
3. $x + y + xy + x^3$ → Yes

Types of Polynomials

1. Polynomials in one variable
   E.g. $x^4 + 1$
2. Polynomials in two variables
   E.g. $x^4 + y^5 + xy$
3. Polynomials in more than 2 variables
   E.g. $xyz + x^2 z^5$

3. $p(x) = x^3 + 2x^2 + x$, $q(x) = x^2 + 2x + 2$

$$\begin{array}{rcl} p(x) & = & x^3 + 2x^2 + x + 0 \\ q(x) & = & 0x^3 + x^2 + 2x + 2 \\ \hline p(x) + q(x) & = & x^3 + 3x^2 + 3x + 2 \end{array}$$

Thus, Let $p(x) = \sum_{k=0}^{n} a_k x^k$ and $q(x) = \sum_{j=0}^{m} b_j x^j$

Then $\boxed{p(x) + q(x) = \sum_{k=0}^{m \vee n} (a_k + b_k) x^k}$

Subtraction Of Polynomials

Ques. Subtract the foll. polynomials :

1. $p(x) = x^2 + 4x + 4$, $q(x) = 10$

$$\begin{array}{rcl} p(x) & = & x^2 + 4x + 4 \\ -q(x) & = & 0x^2 - 0x - 10 \\ \hline p(x) - q(x) & = & x^2 + 4x - 6 \end{array}$$

2. $p(x) = x^4 + 4x$, $q(x) = x^3 + 1$

$$\begin{array}{rcl} p(x) & = & x^4 + 0x^3 + 4x + 0 \\ -q(x) & = & 0x^4 - 1x^3 - 0x - 1 \\ \hline p(x) - q(x) & = & x^4 - x^3 + 4x - 1 \end{array}$$

3. $p(x) = x^3 + 2x^2 + x$, $q(x) = x^2 + 2x + 2$

$$\begin{array}{rcl} p(x) & = & x^3 + 2x^2 + x + 0 \\ -q(x) & = & -0x^3 - x^2 - 2x - 2 \\ \hline p(x) - q(x) & = & x^3 + x^2 - x - 2 \end{array}$$

$\therefore$ Let $p(x) = \sum_{k=0}^{n} a_k x^k$ and $q(x) = \sum_{j=0}^{m} b_j x^j$

Then, $\boxed{p(x) - q(x) = \sum_{k=0}^{m \vee n} (a_k - b_k) x^k}$

Multiplication of Polynomials

Ques. Multiply the foll. polynomials.

1. $p(x) = x^2 + x + 1$ and $q(x) = 2x^3$

Sol^n. $p(x) \cdot q(x) = (x^2 + x + 1)(2x^3)$
$= 2x^5 + 2x^4 + 2x^3$

2. $p(x) = x^2 + x + 1$ and $q(x) = 2x + 1$

$p(x) \cdot q(x) = (x^2 + x + 1)(2x + 1)$
$= 2x^3 + x^2 + 2x^2 + x + 2x + 1$
$= 2x^3 + 3x^2 + 3x + 1$

3. $p(x) = a_2 x^2 + a_1 x + a_0$ & $q(x) = b_1 x + b_0$

Sol^n. $p(x)\cdot q(x) = (a_2x^2 + a_1x + a_0)(b_1x + b_0)$

$= a_2 b_1 x^3 + a_2 b_0 x^2 + a_1 b_1 x^2 + a_1 b_0 x + a_0 b_1 x + a_0 b_0$

$= a_2 b_1 x^3 + (a_1 b_1 + a_2 b_0)x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$

$\therefore$ Let $p(x) = \sum_{k=0}^{m} a_k x^k$ and $q(x) = \sum_{j=0}^{n} b_j x^j$, Then,

$$\boxed{p(x)\cdot q(x) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} (a_j b_{k-j}) x^k}$$

Ques. Multiply $p(x) = x^2 + x + 1$ & $q(x) = x^2 + 2x + 1$.

Let $p(x) = \sum_{k=0}^{n} a_k x^k$ & $q(x) = \sum_{j=0}^{m} b_j x^j$

$p(x)\cdot q(x) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} (a_j b_{k-j}) x^k$

k	a_k	b_j
0	1	1
1	1	2
2	1	1

k	Coefficient	Calculations
0	$a_0 b_0$	1
1	$a_1 b_0 + a_0 b_1$	$1+2=3$
2	$a_0 b_2 + a_1 b_1 + a_2 b_0$	$1+2+1=4$
3	$a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0$	$0+1+2+0=3$
4	$a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0$	$0+0+1+0+0=1$

The resultant polynomial is,

$p(x)\cdot q(x) = x^4 + 3x^3 + 4x^2 + 3x + 1$

Division of Polynomials

- Division of a polynomial by a monomial

$$\frac{3x^2 + 4x + 3}{x} = 3x + 4 + \frac{3}{x}$$

- Division of a polynomial by another polynomial

$$\frac{3x^2 + 4x + 3}{2x + 1} = ??$$

Let us divide, $\frac{3x^2 + 4x + 1}{x + 1}$

$$\Rightarrow \frac{3x^2 + 3x + x + 1}{x + 1} = \frac{3x(x+1) + 1(x+1)}{(x+1)} = \frac{(3x+1)(x+1)}{x+1}$$

$\Rightarrow 3x + 1$ Ans.

Ques. Divide $p(x) = x^4 + 2x^2 + 3x + 2$ by $q(x) = x^2 + x + 1$

$$\frac{p(x)}{q(x)} = \frac{x^4 + 2x^2 + 3x + 2}{x^2 + x + 1} = \frac{x^4 + x^3 + x^2 - x^3 + x^2 + 3x + 2}{x^2 + x + 1}$$

$$= \frac{x^4 + x^3 + x^2}{x^2 + x + 1} \frac{- x^3 + x^2 + 3x + 2}{x^2 + x + 1} = x^2 + \frac{(-x^3 + x^2 + 3x + 2)}{x^2 + x + 1}$$

$$= (x^2) + \left(\frac{-x^3 - x^2 - x}{x^2 + x + 1}\right) + \left(\frac{x^2 + x + x^2 + 3x + 2}{x^2 + x + 1}\right) = x^2 - x + \frac{2x^2 + 4x + 2}{x^2 + x + 1}$$

$$= x^2 - x + \frac{2(x^2+x+x+1)}{x^2+x+1}$$

$$= x^2 - x + \frac{2(x^2+x+1)}{(x^2+x+1)} + \frac{2x}{x^2+x+1}$$

$$= x^2 - x + 2 + \frac{2x}{x^2+x+1}$$ Ans.

Algorithm for Division of Polynomials

Dividend → $\frac{p(x)}{q(x)} = x^2 - x + 2 + \frac{2x}{q(x)}$ ← Remainder

Divisor → $q(x)$; Quotient → $x^2 - x + 2$

Step 1: Arrange the terms in descending order of the degree and add the missing exponents with 0 as coefficient.

Step 2: Divide the first term of the dividend by the first term of the divisor ~~to~~ and get the mono nomial.

Step 3: Multiply the monomial with divisor and subtract the result from the dividend.

Step 4: Check if the resultant polynomial has degree less than divisor. If true, write the remainder else Go to step 2.

Question: Divide : $\frac{2x^3 + 3x^2 + 1}{2x+1}$ by long division method.

$x^2 + x - 1/2$ ← Quotient

Divisor: $2x+1$) $2x^3 + 3x^2 + 1$ ← Dividend

$- \; 2x^3 + x^2$

$2x^2 + 1$

$- \; x + 2x^2 + 0$

$-x + 1$

$- \; -x - 1/2$

$3/2$ ← Remainder

$\frac{2x^3}{2x} = x^2(2x+1)$

$\frac{2x^2}{2x} = x(2x+1)$

$\frac{-x}{2x} = \frac{-1}{2}(2x+1)$

$$\therefore \frac{2x^3 + 3x^2 + 1}{2x+1} = x^2 + x - \frac{1}{2} + \frac{3}{2(2x+1)}$$

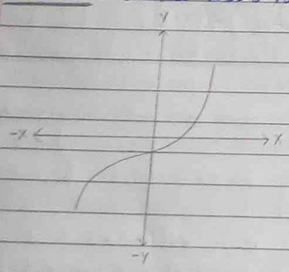
January 20, 2021

WEEK 7

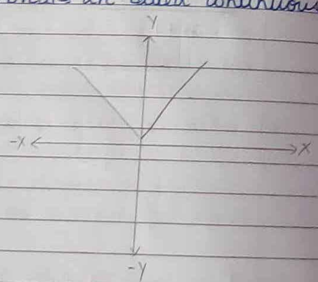
Graph of Polynomials

Characterisation of graphs of Polynomial Functions

- Polynomial functions of second degree or higher have graphs that do not have sharp corners. That is, the graphs are smooth curves.
- Polynomial fn also display graphs that have no breaks. Curves with no breaks are called continuous.



This graph is a graph of polynomial fn.



This is not a graph of polynomial fn.

Zeroes Of Poly. Fn

Zeros of Polynomial Functions

- If f is a polynomial function, the values of x for which $f(x)=0$ are called zeroes of f.
- If the eqn of the polynomial fn can be factored, we can set each factor equal to zero and solve for the zeroes.
- Also any value $x=a$ that is a zero of a poly. fn yields a factor of the polynomial, of the form $(x-a)$.
- Given, the eqn of a polynomial fn, we can use this method to find x-intercepts because at the x-intercepts we find the input values whose output value is zero.
- For general polynomials, this can be a challenging prospect. However, quadratic fn can be solved using the quadratic formula.
- The corresponding formulas for cubic & fourth degree polynomials are not simple enough to remember. And formulas do not exist for general higher degree polynomials.

Factoring

- The polynomial can be factored using known methods:
  a) greatest common factor
  b) factor by grouping

(c) trinomial factoring

- The polynomial is given in factored form.
- Technology is used to determine the intercepts.

x-intercept of Polynomial Function by factoring

1. set $f(x) = 0$
2. If the polynomial f^n is not given in factored form:
   a) Factor out any common monomial factors
   b) factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x-intercepts.

Example: Find x intercept of x^6 $f(x) = x^6 - 8x^4 + 16x^2$

Solution: Set $f(x) = 0$

$\therefore x^6 - 8x^4 + 16x^2 = 0$

$x^2(x^4 - 8x^2 + 16) = 0$

$x^2(x^2 - 4)^2 = 0$

Either $x = 0$ or $x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

$\therefore x = 0, 2, -2$ are the x intercepts of f.

Example: Find x intercepts of $f(x) = x^3 - 4x^2 - 3x + 12$.

Solution: Set $f(x) = 0$

$\therefore x^3 - 4x^2 - 3x + 12 = 0$

$x^2(x-4) - 3(x-4) = 0$

$(x^2-3)(x-4) = 0$

Either $x^2 - 3 = 0$ or $x - 4 = 0$

$x^2 = 3$ or $x = 4$

$x = \pm\sqrt{3}$

Thus, $x = 4, \sqrt{3}, -\sqrt{3}$ are x intercepts of f.

Example: Find the y & x intercepts of $g(x) = (x-1)^2(x+3)$

Sol^n: set $g(x) = 0$

$\therefore (x-1)^2(x+3) = 0$

Either $(x-1)^2 = 0$ or $x + 3 = 0$

$x - 1 = 0$ $\quad$ $x = -3$

$x = 1$

$\therefore x = 1, -3$ are x-intercepts of f.

Now, put $x = 0$ in $g(x)$

$\Rightarrow g(0) = (0-1)^2(0+3) = 1(3) = 3$

Thus, y intercept is 3.

x- intercept of Polynomial Function using Graph.

Find x intercept of $f(x) = x^3 + 4x^2 + x - 6$

In this case, the polynomial is not in a factored form, has no common factors, and does not appear to be factorable using techniques previously discussed.

The only option is to generated the pair of values as done in quadratic case.

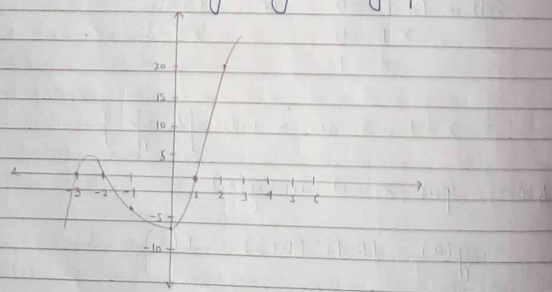
x	y
-2	0
-1	-4
0	-6
1	0
2	20

From table,
$x = -2, 1$ are the x intercepts of f.

The third zero can be found by dividing $f(x)$ by $(x+2)(x-1)$

The third zero of $f(x)$ is $x = -3$

$\therefore$ Join the points smoothly to get the graph



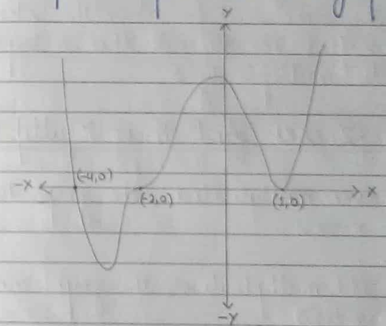
Identification of Zeros & Their Multiplicates

Graphs behave differently at various x-intercepts.

Sometimes, the graph will cross over the horizontal axis at an intercept.

Other times, the graph will touch the horizontal axis and 'bounce off'.

Suppose, for example, we ~~have~~ graph the $f(x) = (x-1)^2(x+2)^3(x+4)$



Identifying zeroes and their multiplicities

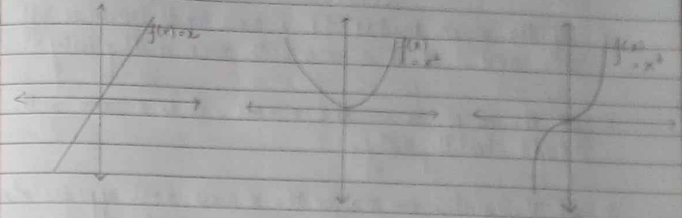
$\Rightarrow$ The x-intercept -4 is the solution of the eq^n $(x+4) = 0$. The graph passes directly through the x-intercept at $x = -4$. The factor is linear (degree 1), so the behaviour near the intercept is like that of a line – it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of f^n.

:) The x-intercept 1 is the repeated solution of the equation $(x-1)^2 = 0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behaviour near the intercept is like that of a quadratic - it bounces of the horizontal axis at the intercept.

:) The x intercept -2 is the repeated solⁿ of the eqⁿ $(x+2)^3 = 0$. The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behaviour near the intercept is like that of a cubic - with the same S shape near the intercept as the toolkit function $f(x) = x^3$. We call this a triple zero, or a zero with multiplicity 3.

Identifying Zeroes And Their Multiplicities

- For zeros with even multiplicities, the graph touch or are tangent to the x-axis.
- For zeros with odd multiplicities, the graphs cross or intersect, the x-axis.
- For higher even powers, such as 4, 6 & 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches & leaves the x-axis.
- For higher odd powers, such as 5, 7, 9... the graph will still cross through the horizontal axis but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.



Graphical Behaviour of Polynomials at X-Intercepts

If a polynomial contains a factor of the form $(x-a)^m$, the behaviour near the x-intercept 'a' is determined by the exponent m. We say that,

$x = a$ is zero of multiplicity m

- The graph of a polynomial fⁿ will touch but not cross the x-axis at zeros with even multiplicities.
- The graph will cross the x-axis at zeros with odd multiplicities.
- The sum of the multiplicities is no greater than the degree of a polynomial fⁿ.

Given, the graph of a polynomial of degree n, how can one identify zeros & their multiplicities?

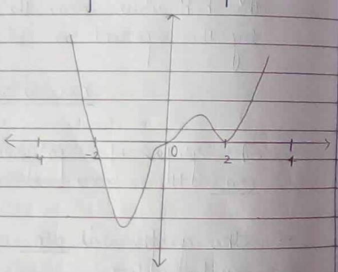
1. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.

2. If the graph crosses the x-axis, it is a zero with odd multiplicity.

3. If the graph ~~cro~~ crosses the x-axis and appears almost linear at the intercept, it is a single zero.

4. The sum of all the multiplicities is no greater than n.

Example : Use the graph of the function of degree 6 to identify the zeros of the fn & their possible multiplicities.

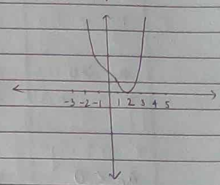


From graph,
$x = -2, 0, 2$

~~(x-2) is linear, deg 1~~
$x = -2$, linear, 1
$x = 0$, odd deg, 3 or 5
$x = 2$, even deg, 2 or 4

sum = 6
$\therefore$ $x = 0$ is with multiplicity 3
$x = 2$ is with multiplicity 2
$x = -2$ with multiplicity 1.

Example : Use the graph of the fn of degree 4 to identify the zeroes of the fn and their possible multiplicities.



$x = 2$

$x = 2$, even degree, 2 or 4

Hence the fn must have a factor $(x-2)^2$

End Behaviour Of Poly.

As, we have already observed, the behaviour of a polynomial function,

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

is either increasing or decreasing as the value of x increases which is mainly due to the fact that the leading term dominate the behaviour of polynomial. This behaviour is known as End Behaviour of fn.

As observed in quad. eqns, if the leading term of a polynomial function, $a_n x^n$, is an even power function and $a_n > 0$, then as x increases or decreases, f(x) increases and is unbounded.
When the leading term is an odd power fn, as x decreases, f(x) also decreases and is unbounded ;

as x increases, f(x) also increases and is unbounded.

	EVEN DEGREE	ODD DEGREE
$a_n > 0$	$x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to \infty$	$x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to -\infty$
$a_n < 0$	$x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to -\infty$	$x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to \infty$

Relationship Between the Degree & Turning Points

As seen in quadratic case, a polynomial of degree 2 has one turning point.

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising).

We can hypothesize that a polynomial of degree n can have at most (n-1) turning points.

Find the maximum possible no. of turning points of each polynomial function.

1. $f(x) = 1 + x^2 + 4x^5$
   $n = 5$
   $\therefore$ Turning pts $= n-1 = 5-1 = 4$ (max. possible)

2. $f(x) = (x-1)^3(x+2)$
   $n = 4$
   $\therefore$ Turning pts $= n-1 = 4-1 = 3$ (max. possible)

Graphing a Poly. F^n

1. Find the x- and y- intercepts if possible.
2. Check for symmetry. If the f^n is an even f^n, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a f^n is an odd function, its graph is symmetrical about the origin, that is $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behaviour of the polynomial at the x-intercepts.
4. Determine the end behaviour by examining the leading ~~term~~ term.
5. Use the end behaviour and the behaviour at the intercepts to sketch a graph.

6. Ensure that the no. of turning points does not exceed one less than the degree of the polynomial.

7. Optionally, use graphing tools to check the graph.

Example : Sketch a graph of $f(x) = -(x+2)^2(x-5)$

x- intercepts are $x = -2, 5$

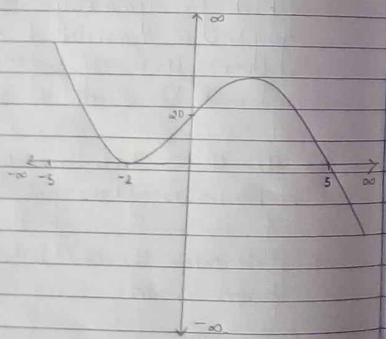
$x = -2$ has multiplicity 2, quadratic
$x = 5$ has multiplicity 1, linear

y intercept $f(0) = 20$.

The leading term is $-x^3$. Therefore the odd degree polynomial with negative leading coefficient has the following end behaviour.

$x \to \infty$, $f(x) \to -\infty$
$x \to -\infty$, $f(x) \to \infty$

f can have at most
$3-1 = 2$ turning pts.



INTERMEDIATE VALUE THEOREM

Let f be a polynomial function. The Intermediate Value Theorem states that if f(a) and f(b) have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

DERIVING FORMULA FOR POLYNOMIAL F

Given the graph, how to find the formula for polynomial functions ?

1. Find the x- intercepts of the graph to find the factors of the polynomial.

2. Understand the behaviour of the graph at the x-intercepts to determine the multiplicity of each factor.

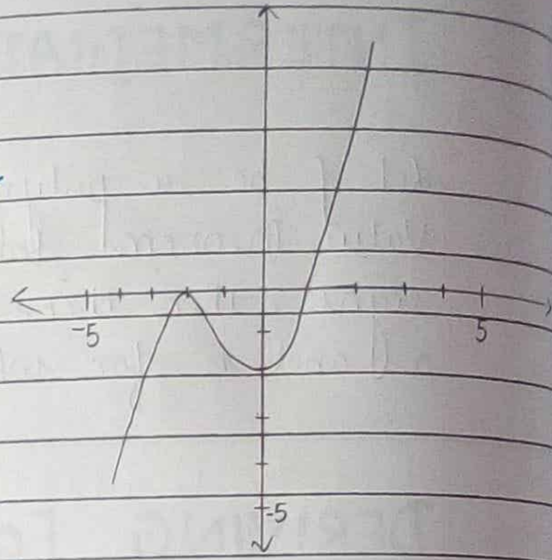
3. Find the polynomial of least degree containing all the factors found in the previous step.

4. Use any other point on the graph (the y-intercept may be easiest) to determine the stretch factor.

Example : Write the formula for polynomial given in the graph.

$x = -2, 1$ are the x-intercepts and the fn has 2 turning points. The end behaviour is similar to odd degree polynomial with +ve leading term, i.e. it may be a polynomial of degree 3.

The behaviour at $x=1$ is linear and $x=-2$ is of even degree and hence quadratic.
The resultant polynomial is of degree 3 withe zeroes -2 and $+1$ with multiplicities 2 and 1 resp.



The polynomial has form, $f(x) = a(x+2)^2(x-1)$

To determine a, use y- intercept.
From graph, $f(0) = -2$
From the form, $f(0) = -4a$
$\Rightarrow a = \frac{1}{2}$

Hence, the f^n must be, $f(x) = \frac{1}{2}(x+2)^2(x-1)$

Date: February 1, 2021

WEEK 8

Exponential Functions

1. ONE-TO-ONE FUNCTIONS

A function $f: A \to B$ is called one-to-one, if for any $x_1 \neq x_2 \in A$, then $f(x_1) \neq f(x_2)$. Thus, If $f(x_1) = f(x_2)$ $\Rightarrow$ $x_1 = x_2$

The Horizontal line test is used to find whether the given fn is one-to-one or not.

One-to-one fn never fails horizontal line test.

One-to-one fn are ~~not~~ always reversible on their range.

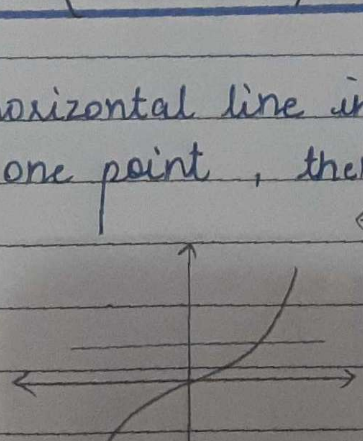
If a fn fails the horizontal line test, then it is not reversible.

Examples: $f(x) = x$; $f(x) = x^3$

Theorem (The Horizontal Line Test)

If any horizontal line intersects the graph of a fn f in at most one point, then f is one-to-one.

PROOF: $f(x) = x^3$

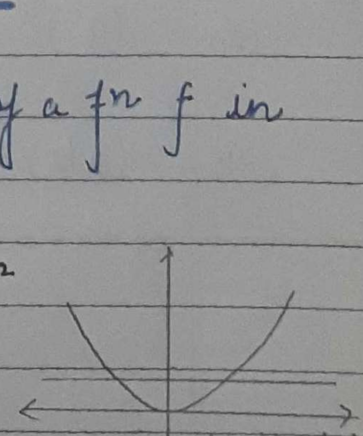


[Let us pass horizontal line, $y = 2$]

Test Pass.

horizontal line passes through only one pt.

$f(x) = x^2$



Test Fail

horizontal line passes through more than 1 pt.

Question : Can we identify the class of functions that are one to one ?

For every $x_1, x_2 \in A$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
→ Increasing Functions.

For every $x_1, x_2 \in A$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
→ Decreasing Functions

Thus, Increasing & Decreasing f^ns are one-to-one.

EXPONENTS

a^r, where $a > 0$ and $r \in \mathbb{R}$

Here, a is base and r is exponent.

Laws of Exponents

For $s, t \in \mathbb{R}$ and $a, b > 0$

Recall, $1^s = 1$; $a^{-s} = \frac{1}{a^s}$; $a^0 = 1$, $a > 0$

(i) $a^s \cdot a^t = a^{s+t}$

(ii) $(a^s)^t = a^{st}$

(iii) $(ab)^s = a^s b^s$

---

2. EXPONENTIAL FUNCTIONS

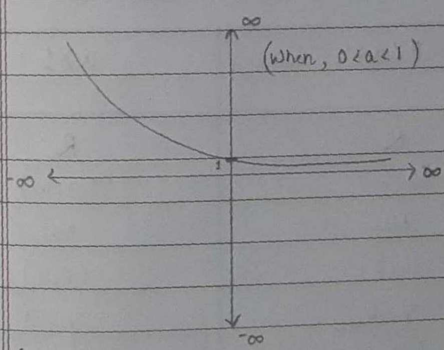
An exponential function in standard form is given by $\boxed{f(x) = a^x}$, where $a > 0$ and $a \neq 1$.

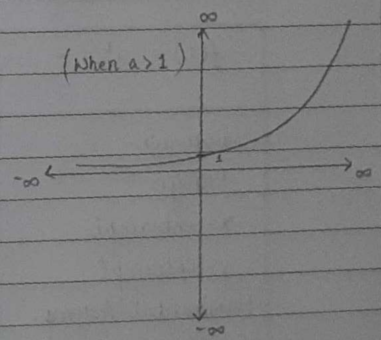
Observations.

1. Domain of f is $\mathbb{R}$, i.e, $(-\infty, \infty)$
2. Range of f is $\mathbb{R}^+$, i.e, $(0, \infty)$

Why $a \neq 1$?
because, if $a = 1$, ie, $1^x = 1$ (constant f^n)
then it will not remain an exponential function.

GRAPH OF EXPONENTIAL FUNCTIONS





Example, $(1/3)^x$, $(1/5)^x$, $(2/8)^x$, etc

Examples, 2^x, 3^x, 5^x etc.

Properties : $f(x) = 2^x$
→ $\text{Domain}_f = (-\infty, \infty)$
→ $\text{Range}_f = (0, \infty)$
→ y intercept = $(0, 1)$
→ x intercept = NIL
→ $y = 0$ Horizontal Asymptote

→ End Behaviour :
$x \to \infty$, $2^x \to \infty$
$x \to -\infty$, $2^x \to 0$
→ No roots
→ Increasing f^n.

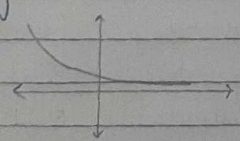
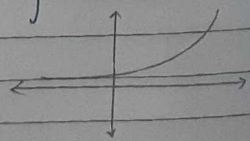
Fact : Every $f(x) = a^x$, $a>1$ has same properties as 2^x.

Properties [$f(x) = 2^{-x} \Rightarrow f(x) = (1/2)^x$] $0<a<1$

- → Domain$_f$ = $\mathbb{R}$ → Range$_f$ = $(0, \infty)$
- → y intercept = $(0,1)$ → x intercept = NIL (no roots)
- → End Behaviour,
  - $x \to \infty$, $(1/2)^x \to 0$
  - $x \to -\infty$ $(1/2)^x \to \infty$
- → It is a decreasing f^n.

Fact : Every $f(x) = a^x$, $0<a<1$ has same properties as $\left(\frac{1}{2}\right)^x$.

SUMMARY

$f(x) = a^x$	$0<a<1$	$a>1$
Domain	$\mathbb{R}$	$\mathbb{R}$
Range	$(0, \infty)$	$(0, \infty)$
x- intercept	NIL	NIL
y- intercept	$(0,1)$	$(0,1)$
Horizontal Asymp.	$y=0$	$y=0$
Inc. / Dec. f^n	Decreasing	Increasing
End Behaviour		
$x \to \infty$	$f(x) = a^x \to 0$	$f(x) = a^x \to \infty$
$x \to -\infty$	$f(x) = a^x \to \infty$	$f(x) = a^x \to 0$
Graphs		

3. NATURAL EXPONENTIAL F^N

- :> From the theory of limits, it is known that

$$\left(1 + \frac{1}{n}\right)^n \longrightarrow e \quad \text{as } n \longrightarrow \infty$$

- :> Existence of 'e' is studied in calculus.
- :> e is irrational no.

$\boxed{e \to \text{Euler's no.}}$

- :> $e \approx 2.71828\ldots$

Question : Why is 'e' so important?
→ Interest Rate Calculation
Continuous compounding

Example, Invested = ₹1 Interest = 1%

$$\therefore \left(1 + \frac{0.01}{12}\right)^{12}$$ [If bank revises every month for one year]

Thus, $1 \times e^{0.01t}$ → where t is time

Generalising, $\left(1 + \frac{x}{n}\right)^{nt}$, where t is time, x is interest rate.

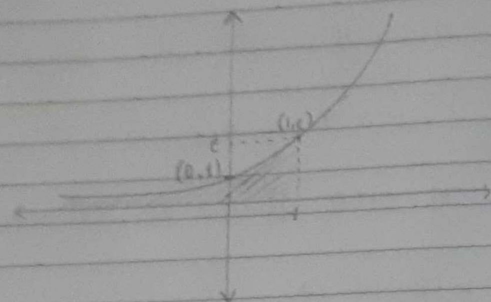
$= e^{xt}$

Definition : The natural exponential function is defined as $f(x) = e^x$

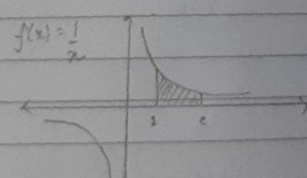
PROPERTIES Domain of f = $\mathbb{R}$
Range of f = $(0, \infty)$

And we know, $e > 1$.

GRAPH :



- e is the slope of the tangent line to $f(x) = e^x$ at $(1, e)$.
- The area under the $f(x) = e^x$ from $(-\infty, 1)$ is e.
- For $f(x) = \frac{1}{x}$, $x \in (1, e)$, the area under the curve is 1 (a unit).



EXAMPLE : Let R be the percent of people who respond to affiliate links under YouTube descriptions & purchase the product in 't' minutes is given by

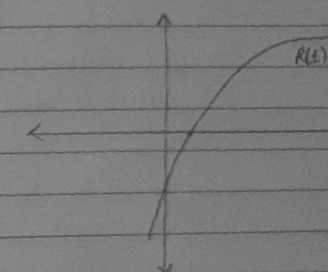
$$R(t) = 50 - 100\,e^{-0.2t}$$

(a) What is the %age of people responding after 10 min.?
(b) What is the highest percent expected?
(c) How long before R(t) exceeds 30%.!

Solution : (a) At $t = 10$,

$$\therefore R(10) = 50 - 100\,e^{-0.2 \times 10}$$
$$= 50 - 100\,e^{-2}$$
$$= 50 - \frac{100}{e^2}$$
$$\Rightarrow R(10) = 36.46\,\%$$

(b) $R(t) = 50 - 100\,e^{-0.02t}$



Answer : 50%.

(c) $R(t) = 50 - 100\,e^{-0.02t}$
$\Rightarrow 30 = 50 - 100\,e^{-0.02t}$
$\Rightarrow 100\,e^{-0.02t} = 20$
$\Rightarrow e^{-0.02t} = \frac{1}{5}$ $\quad \ln\left(\frac{1}{5}\right) = -0.02t$

$t \approx 8$ min (from graph)

Domain Of Composite Functions. [$f(x)$, $g(x)$, $fog(x)$]

Step 1 : Find domain of g
Step 2 : Find range of g
Step 3 : Find domain of f.

CASE 1 → If Range of $g \subseteq$ Domain of f
$\Rightarrow$ Domain of (fog) = Domain of g

CASE 2 → If Range of $g \not\subseteq$ Domain of f, then
We have to eliminate the elements from the domain of g for which we are getting those elements in Range(g) which are not in Domain(f).

Date
February 1, 2021

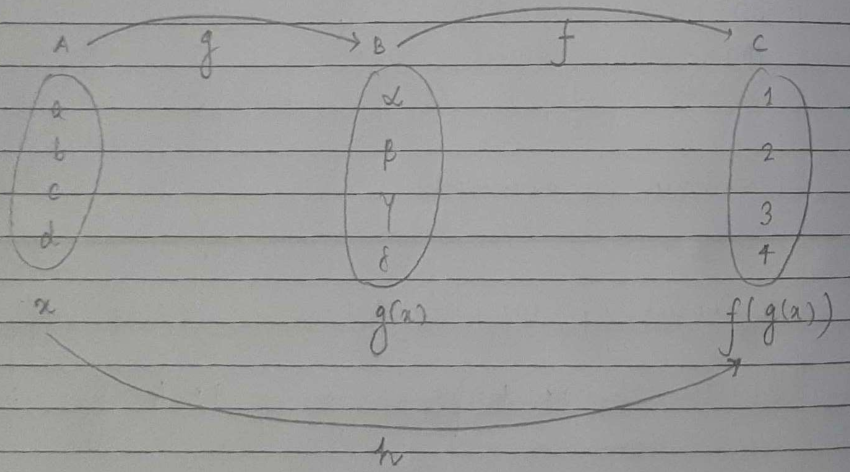
Composite Functions

The composition of functions f & g is denoted by fog & is defined by

$$fog(x) = f(g(x))$$

→ The domain of the composite fn fog is the set of all x such that

- x is in the domain of g
- g(x) is in the domain of f.



∴ $g : A \to B$ — Domain : values of x in A

$f : B \to C$ — Domain : values of g(x) in B

⇒ $h : A \to C$ — Domain : values of x in A, i.e. D_g

$h(x) = f(g(x))$ — Range : values of f(g(x)) in C, i.e. R_f.

The domain of a composite fn, fog(x) is always a subset of the domain of the fn g(x).

The range of a composite fn fog(x) is always a subset of the range of the fn f(x).

Examples : Given $f(x) = 3x - 4$

$g(x) = x^2$.

Find : (i) $fog(x)$ (ii) $gof(x)$

(i) $(fog)(x) = f[g(x)] = f[x^2] = 3x^2 - 4$

(ii) $(gof)(x) = g[f(x)] = g[3x-4] = (3x-4)^2$

Q1. Given : $f(x) = x + 1$, $g(x) = x^2 - 1$. Find :
(i) $gof(x)$ (ii) $fog(x)$

Sol. (i) $gof(x) = g[f(x)] = g[x+1] = (x+1)^2 - 1 = x^2 + 2x$

$x^2 + 1 + 2x$

(ii) $fog(x) = f[g(x)] = f[x^2 - 1] = x^2 - 1 + 1 = x^2$

Q2. Given, $f(x) = 10(x-5)^2 + 100x - 225$ and $g(x) = \sqrt{2x+5}$, find $fog(x)$ and $gof(x)$.

Sol. $fog(x) = f[\sqrt{2x+5}] = 10(\sqrt{2x+5} - 5)^2 + 100(\sqrt{2x+5}) - 225 = 10[(2x+5) + 25 - 10\sqrt{2x+5}] + 100\sqrt{2x+5} - 225$

⇒ $fog(x) = 20x + 50 + 250 - 100\sqrt{2x+5} + 100\sqrt{2x+5} - 225 = 20x + 300 - 225$

$= 20x + 75$

$$gof(x) = g[10(x-5)^2 + 100x - 225]$$

$$= \sqrt{2(10x^2 + 250 - \cancel{100x} + \cancel{100x} - 225)} + 5$$

$$= \sqrt{20x^2 + 50 + 5} = \sqrt{20x^2 + 55}$$

Thus, $fog(x) = 20x + 75$ & $gof(x) = \sqrt{20x^2 + 55}$

Q3. Suppose $f(x) = 3x + 10$ & $g(x) = \sqrt{x+11}$ are two f^n. Find the value of $f(g(5))$. and $g[f(5)]$.

Sol.3. $fog(5) = f[g(5)] = f[\sqrt{5+11}] = f[\sqrt{16}] = f[(\pm 4)]$ Only the value accepted

$f(4) = 3(4) + 10 = 22$ ✓ D_f (square root f^n) $= [0, \infty)$

$f(-4) = -12 + 10 = -2$ ✗

Now, $gof(5) = g[f(5)] = g[15 + 10] = g[25]$

$\Rightarrow g(25) = \sqrt{25 + 11} = \sqrt{36} = 6$

Q4. Give examples, where $fog(x) = gof(x)$.

Sol.4. (i) $f(x) = 7x + 6$ and $g(x) = 4x + 3$

(ii) $f(x) = e^x$ and $g(x) = x$

Determination of Domain for C. f^n.

The following values must be excluded from input x.

\# If there exists $x \in R$ that is not in the domain of a f^n g, then that x will not be in the domain of some composite f^n (fog).

- if $x \notin D_g \Rightarrow x \notin Dom(fog)$
- $\{x \mid g(x) \notin D_f\}$ must not be included in $Dom(fog)$.

Example: $f(x) = \frac{2}{x-1}$, $g(x) = \frac{3}{x}$

Find $fog(x)$ & Domain (fog).

Sol. $fog(x) = f[g(x)] = f\left[\frac{3}{x}\right] = \frac{2}{3/x - 1} = \frac{2x}{3-x}$

$D_f = R - \{1\}$

$D_g = R - \{0\}$ $D_{fog} = R - \{3\}$ from here

Thus, Overall : Domain $(fog) = R - \{0, 3\}$

Q5. $f(x) = \frac{1}{x+1}$, $g(x) = \frac{1}{x}$. Find $fog(x)$ & $D(fog)$.

Sol.5. $fog(x) = f\left[\frac{1}{x}\right] = \frac{1}{1/x + 1} = \frac{x}{x+1}$

Rule 1: $D_g = R - \{0\}$

Rule 2: $D_f = R - \{-1\}$ $\therefore g(x) = -1 \Rightarrow 1/x = -1 \Rightarrow x = -1$

$\therefore$ Domain $(fog) = R - \{0, -1\}$

\# The domain of a composite f^n (fog) is the set of all x such that x is in the domain of f^n g and $g(x)$ is in the domain of a f^n f.

Inverse Functions

The inverse of a function f, f^{-1} is a function such that

$$f^{-1}(f(x)) = x \quad \forall\, x \in D_f = \text{Range}(f^{-1})$$

$$\& \quad f(f^{-1}(x)) = x \quad \forall\, x \in D_{f^{-1}} = \text{Range}(f)$$

\# f is one-to-one f^n.
\# All one-to-one f^n are reversible.

Example 1.: $g(x) = x^3$ & $g^{-1}(x) = \sqrt[3]{x}$.
Verify that they are inverse of each other.

Sol.

$$g \circ g^{-1}(x) = g[g^{-1}(x)] = g[\sqrt[3]{x}] = g(x^{1/3}) = (x^{1/3})^3 = x$$

$$g^{-1} \circ g(x) = g^{-1}[g(x)] = g^{-1}(x^3) = (x^3)^{1/3} = x$$

Hence verified.

Example 2: Verify f is inverse of g, where $f(x) = \dfrac{x-5}{2x+3}$ and $g(x) = \dfrac{3x+5}{1-2x}$

Sol. Let, $f(x) = y \Rightarrow y = \dfrac{x-5}{2x+3} \Rightarrow (2x+3)y = x-5 \Rightarrow 2xy + 3y = x-5$

$\Rightarrow 3y + 5 = x - 2xy$

$\Rightarrow 3y + 5 = x(1-2y)$

$\Rightarrow x = \dfrac{3y+5}{1-2y} = g(y)$

$\therefore g(x) = f^{-1}(x)$
or $f(x) = g^{-1}(x)$

Other Method: $f \circ g(x) = f[g(x)] = f\left[\dfrac{3x+5}{1-2x}\right]$

$$\Rightarrow f \circ g(x) = \frac{\frac{3x+5}{1-2x} - 5}{2\left(\frac{3x+5}{1-2x}\right) + 3} = \frac{3x+5-5(1-2x)}{6x+10+3(1-2x)}$$

$$\Rightarrow f \circ g(x) = \frac{3x+\cancel{5}-\cancel{5}+10x}{\cancel{6x}+10+3-\cancel{6x}} = \frac{13x}{13} = x$$

Now, $g \circ f(x) = g[f(x)] = g\left[\dfrac{x-5}{2x+3}\right]$

$$\Rightarrow g \circ f(x) = \frac{3\left(\frac{x-5}{2x+3}\right)+5}{1-2\left(\frac{x-5}{2x+3}\right)} = \frac{3x-\cancel{15}+10x+\cancel{15}}{\cancel{2x}+3-\cancel{2x}+10} = \frac{13x}{13} = x$$

$\therefore f \circ g(x) = g \circ f(x) = I(x) = x$

Hence Verified!

Graph of f & f^{-1}

$$x = a \underset{f^{-1}}{\overset{f}{\rightleftarrows}} f(a) = y$$

Values of x changes to y and vice versa.

If $(a, f(a))$ is on the graph of f, then $(f(a), a)$ is on the graph of f^{-1}.

\# Theorem: The graph of f & f^{-1} are symmetric across $y = x$ line

Date: February 8, 2021

WEEK 9

Logarithmic Functions

Recall, $f(x) = a^x$ $(a>0,\ a\neq 1)$ is one-to-one f^n, thus it has an inverse f^n of it.

<u>Definition</u>: The logarithmic function (to the base a) in standard form is

$$Y = \log_a(x)$$

and is defined to be the inverse of $f(x) = a^x$.

$$\boxed{Y = \log_a x \iff x = a^Y}$$

$a^{\log_a x} = x$ and $\log_a a^x = x$

$$\text{Domain}(\log_a x) = \text{Range}(a^x) = (0,\infty)$$

$$\text{Range}(\log_a x) = \text{Domain}(a^x) = \mathbb{R}$$

Example: Find the ~~range~~ domain of $f(x) = \log_4(1-x)$

$$\text{Domain}(\log_4(1-x)) = \text{Range}(4^{1-x}) = (-\infty, 1)$$

Example: Find the domain of $g(x) = \log_3\left(\frac{1+x}{1-x}\right)$

We know, $\text{Domain}(\log_3 x) = (0,\infty)$

i.e, $x \in (0,\infty)$

$\Rightarrow \frac{1+x}{1-x} > 0$

$\Rightarrow 1+x > 0$ & $1-x > 0 \Rightarrow x < 1$

$\Rightarrow x > -1$, Also $x \neq 1$

$\therefore$ Domain $= (-1, 1)$ ~~$\cup (1,\infty)$~~

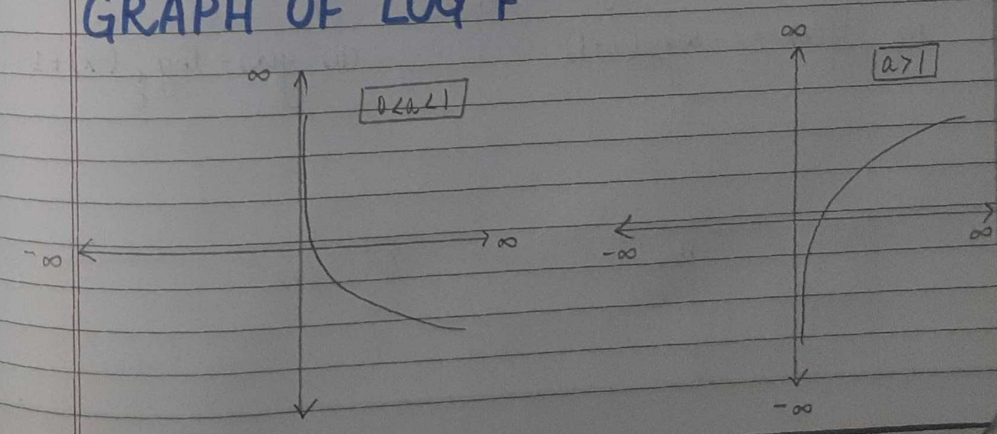
If, $a^u = a^v$ $(a>0,\ a\neq 1)$

$\Rightarrow u = v$

Question: Find $\log_3\left(\frac{1}{9}\right)$

$\rightarrow \log_3\left(\frac{1}{9}\right) = \log_3\left(3^{-2}\right) = -2$ Ans.

GRAPH OF LOG F^N



Feb 13, 2021

PROPERTIES OF GRAPH OF LOG F^N

Properties of $f(x) = \log_a(x)$

→ $\text{Domain}_f = (0, \infty)$

→ $\text{Range}_f = (-\infty, \infty) = \mathbb{R}$

→ x-intercept = $(1,0)$

→ y-intercept = NIL

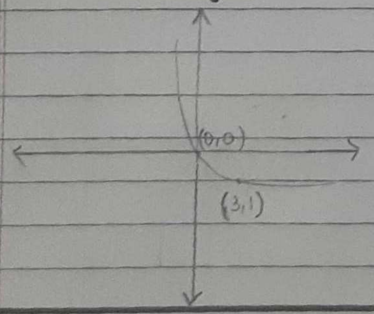
→ Vertical Asymptote at $x = 0$ (y axis)

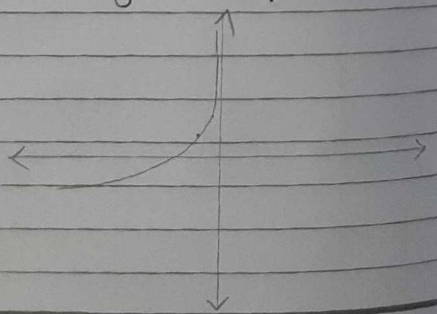
→ f is one to one & passes through $(1,0)$ & $(a,1)$

→ When, $0<a<1$ → f^n is decreasing
  and, $a>1$ → f^n is increasing

EXAMPLE : Draw graphs of following:

(i) $f(x) = -\log_4(x+1)$ (ii) $g(x) = \log_{\frac{1}{4}}(-x)+1$





NATURAL LOGARITHMIC F^N

The natural logarithmic function is $f(x) = \log_e(x)$ where the base is 'e'.

It is always denoted by $\ln(x)$, i.e,
$f(x) = \ln(x)$

REMARK : $\ln(e^x) = x$, $\forall\ x \in \mathbb{R} = \text{Dom}(e^x)$
$e^{\ln x} = x$, $\forall\ x \in (0,\infty) = \text{Dom}(\ln x)$

COMMON LOG (BASE 10) $\log x = \log_{10} x$

SOLVING EXPONENTIAL $EQ^{N}s$.

1. Solve for x : $2^{x+1} = 64$

→ $x+1 = \log_2 64 \Rightarrow x+1 = 6 \Rightarrow \boxed{x=5}$

2. Solve for x: $e^{-x^2} = (e^x)^2 \frac{1}{e^3}$

→ $\frac{1}{e^{x^2}} = (e^x)^2 \frac{1}{e^3} \Rightarrow e^{-x^2} = e^{2x-3}$
$\Rightarrow -x^2 = 2x-3$
$\Rightarrow x^2+2x-3 = 0$
$\Rightarrow x^2+3x-x-3 = 0$
$\Rightarrow (x+3)(x-1) = 0$; $x = 1, -3$

3. Solve for x : $9^x - 2.3^{x+1} - 27 = 0$

Sol. $9^x - 2(3^{x+1}) - 27 = 0$

$\Rightarrow 3^{2x} - 2\cdot 3^x\cdot 3 - 27 = 0$

$\Rightarrow (3^x)^2 - 6\cdot(3^x) - 27 = 0$

Let $3^x = y$

$\therefore y^2 - 6y - 27 = 0$

$\Rightarrow y^2 - 9y + 3y - 27 = 0$

$\Rightarrow y(y-9) + 3(y-9) = 0$

$\Rightarrow (y+3)(y-9) = 0$

$\therefore$ When $y = -3$

$\Rightarrow 3^x = -3$

not possible

When $y = 9$

$\Rightarrow 3^x = 9$

$\Rightarrow \boxed{x = 2}$

4. Solve for x : $5^{x-2} = 3^{3x+2}$

Sol. Taking log both sides, $\log 5^{(x-2)} = \log 3^{(3x+2)}$

$\Rightarrow (x-2)\log 5 = (3x+2)\log 3$

$\Rightarrow x\log 5 - 2\log 5 = 3x\log 3 + 2\log 3$

$\Rightarrow 3x\log 3 - x\log 5 = -2\log 3 - 2\log 5$

$\Rightarrow x[3\log 3 - \log 5] = -2[\log 3 + \log 5]$

$\Rightarrow x[\log 27 - \log 5] = -2\log 15$

$\Rightarrow x = \dfrac{-2\log 15}{\log\left(\frac{27}{5}\right)} = \dfrac{\log(1/225)}{\log(27/5)}$ Ans.

5. Solve : $x + e^x = 2$

$\rightarrow x + e^x = 2 \Rightarrow e^x = 2 - x \Rightarrow x = \ln(2-x)$

$\Rightarrow \ln(2-x) - x = 0$

$x = 0.443$ [by graph]

PROPERTIES OF LOG F^N

1. $\log_a 1 = 0$ $\quad a \in (0,1)\ ;\ a > 1$

2. $\log_a a = 1$

3. $a^{\log_a x} = \log_a(a^x) = x$

LAWS OF LOGARITHM

Let $r \in \mathbb{R}$, $0 < a < 1$ or $a > 1$; $M, N > 0$

Then,

1. $\log_a(MN) = \log_a M + \log_a N$

2. $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

3. $\log_a\left(\frac{1}{N}\right) = -\log_a N$

4. $\log_a(M^r) = r\log_a M$

5. $\log_{a^r}(M) = \frac{1}{r}\log_a M$

6. $\log_a x = \dfrac{1}{\log_x a}$

Application of Laws of Logarithm

1. Simplify : $\log_a\left(\frac{x^3\sqrt{x^2+1}}{(x+3)^4}\right)$

Sol.1 $\log_a x^3 + \log_a \sqrt{x^2+1} - \log_a (x+3)^4$

$= 3\log_a x + \frac{1}{2}\log_a (x^2+1) - 4\log_a (x+3)$

2. Combine using logs : $2\log_a x + \log_a 9 + \log_a (x^2+1) - \log_a 5$

Sol.2 $\log_a x^2 + \log_a 9 + \log_a (x^2+1) - \log_a 5$

$= \log_a\left[\frac{x^2(9)(x^2+1)}{5}\right] = \log_a\left[\frac{9x^4+9x^2}{5}\right]$

CHANGE OF BASE RULE

If $0<a<1$ or $a>1$ and $0<b<1$ or $b>1$

Then, for $x>0$, $\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$

Now, $\log_e x = \frac{\log_{10} x}{\log_{10} e} = \log_{10} x \cdot \log_e 10 = 2.303 \log_{10} x$

Thus, $\boxed{\ln x = 2.303 \log x}$

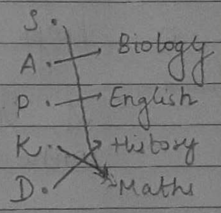
Dat February 13, 2021

WEEK 10

Introduction to Graphs

VISUALIZING RELATIONS

- Cartesian Product $A \times B$
  $\{(a,b) \mid a \in A, b \in B\}$
- A relation is a subset of $A \times B$
- Teachers and Courses
  - → T, set of teachers in a college
    C, set of courses being offered
  - → $A \subseteq T \times C$ describes the allocation of teachers to courses
  - → $A = \{(t,c) \mid (t,c) \in T \times C,\ t \text{ teaches } c\}$



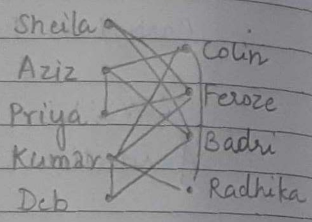
GRAPHS

- Graph : $G = (V, E)$
  - → V is a set of vertices or nodes
    - → One vertex, many vertices
  - → E is the set of edges
  - → $E \subseteq V \times V$ – binary relation
- Directed Graph
  - → $(v, v') \in E$ does not imply $(v', v) \in E$

→ The teacher course graph is directed.

- Undirected Graph
  - → $(v, v') \in E$ iff $(v', v) \in E$
  - → Effectively (v, v'), (v', v) are the same edge
  - → Friendship relation

Friendship as a graph



PATHS

- Priya needs some help that Radhika can provide. How will Priya come to know about this?

Priya – Aziz – Badri – Radhika

or Priya – Feroze – Kumar – Radhika

- A path is a sequence of vertices $v_1, v_2, \ldots, v_k$ connected by edges.
  - → For $1 \le i < k$, $(v_i, v_{i+1}) \in E$

- Normally, a path does not visit a vertex twice
  - → Kumar – Feroze – Colin – Aziz – Priya – Feroze – Sheila
  - → Such a sequence is usually called a '<u>walk</u>'.

REACHABILITY

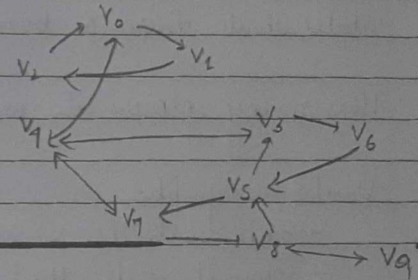
- Paths in directed graphs

- How can I fly from Madurai to Delhi?
  - → Find a path from V_9 to V_0.

- Vertex v is reachable from vertex u if there is a path from u to v.

- Typical Questions
  - → Is v reachable from u?
  - → What is the shortest path from u to v?
  - → What are the vertices reachable from u?
  - → Is the graph connected? Are all vertices reachable from each other?

AIRLINE ROUTES →



SUMMARY

- A graph represents relationships between entities
  - → Entities are vertices / nodes
  - → Relationships are edges

- A graph may be directed or undirected
  - → A is a parent of B – directed
  - → A is a friend of B – undirected

- Paths are sequences of connected edges

- Reachability: is there a path from u to v?

Some General Graph Problems

MAP COLORING

- Assign each state a colour
- States that share a border should be coloured differently
- How many colours do we need ?
- Create a graph
  - → Each state is a vertex
  - → connect states that share a border
- Assign colours to nodes so that endpoints of an edge have different colours
- Only need the underlying graph
- Abstraction : if we distort the graph problem is unchanged .

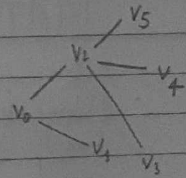
GRAPH COLOURING

- Graph $G = (V, E)$, set of colours C
- Colouring is a f^n $c : V \to C$ such that $(u, v) \in E \Rightarrow c(u) \neq c(v)$

- Given $G = (V, E)$, what is the smallest set of colours need to colour G
  - → 'Four Colour Theorem' for graphs derived from geographical maps , 4 colours suffice
  - → Not all graphs are planar . General case ? Why do we care ?
- How many classrooms do we need ?
  - → Courses and timetable slots
  - → Graph : Edges are overlaps in slots
  - → Colours are classrooms

VERTEX COVER

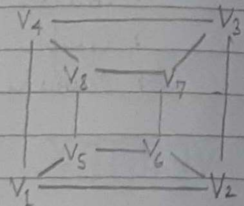
- A hotel wants to install security cameras
  - → All corridors are straight lines
  - → camera at the intersection of corridors can monitor all those corridor .
- Minimum no. of cameras needed ?



- Represent the floor plan as a graph
  - → V - intersections of corridors
  - → E - corridor segments connecting intersections
- Vertex cover
  - → Marking v covers all edges from v
  - → Mark smallest subset of V to cover all edges .

INDEPENDENT SET

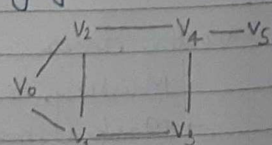
- A dance school puts up group dances
  - → Each dance has a set of dancers
  - → Sets of dancers may overlap across dances
- Organizing a cultural programme
  - → Each dancer performs at most once
  - → Max. no. of dances possible!



- Represent the dances as a graph
  - → V - dances
  - → E - sets of dancers overlap
- Independent set
  - → Subset of vertices such that no two are connected by an edge.

MATCHING

- Class project can be done by one or two people
  - → If two people, they must be friends
- Assume we have a graph describing friendships
- Find a good allocation of groups



- Matching
  - → $G = (V,E)$, an undirected graph
  - → A matching is a subset $M \subseteq E$ of mutually disjoint edges
- Find a maximal matching in G
- Is there a perfect matching, covering all vertices?

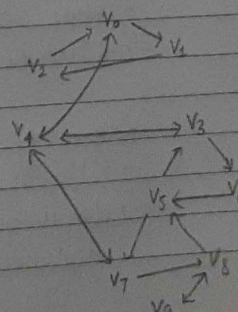
SUMMARY

- Graphs are useful abstract representations for a wide range of problems
- Reachability and connectedness are not the only interesting problems we can solve on graphs
  - → Graph colouring
  - → Vertex cover
  - → Independent set
  - → Matching
  - ....

Working with Graphs

AIRLINE ROUTES

- Looking at the picture of G, we can "see" that v_0 is reachable from v_9
- How do we represent this picture so that we can compute reachability?



ADJACENCY MATRIX

- Let $|V| = n$
  - → assume $V = \{0, 1, \dots, n-1\}$
  - → use a table to map actual vertex "names" to this set

- Edges are now pairs (i, j), where $0 \le i, j < n$
  - → usually assume $i \neq j$, no self loops

- Adjacency matrix
  - → Rows and columns numbered $\{0, 1, \dots, n-1\}$
  - → $A[i,j] = 1$, if $(i,j) \in E$

- Undirected graph
  - → $A[i,j] = 1$ iff $A[j,i] = 1$
  - → Symmetric across main diagonal

COMPUTING WITH ADJACENCY MAT.

- Neighbours of i - column j with entry 1
  - → scan row i to identify neighbours of i
  - → Neighbours of 6 are 3 and 5

- Directed graph
  - → rows represent outgoing edges
  - → columns represent incoming edges

- Degree of a vertex i
  - → No. of edges incident on i, degree(6) = 2
  - → For directed graphs, outdegree & indegree
    indegree(6) = 1, outdegree(6) = 1

CHECKING REACHABILITY

- Is Delhi (0) reachable from Madurai (9)?

- Mark 9 as reachable

- Mark each neighbour of 9 as reachable

- Systematically mark neighbours of marked vertices

- Stop when 0 becomes marked

- If marking process stops without target becoming marked the target is unreachable

- Mark source vertex as reachable

- Systematically mark neighbours of marked vertices

- Stop when target becomes marked

- Need a strategy to systematically explore marked neighbours

- Two primary strategies
  - → Breadth first: propagate marks in 'layers'
  - → Depth first: explore a path till it dies out, then backtrack

ADJACENCY LISTS

- Adjacency matrix has many 0's
  - → size is n^2, regardless of no. of edges
  - → undirected graph : $|E| \le \frac{n(n-1)}{2}$
  - → directed graph : $|E| \le n(n-1)$
  - → Typically $|E|$ is much less than n^2
- Adjacency list
  - → List of neighbours for each vertex

0	{1,4}	5	{3,7}
1	{2}	6	{5}
2	{0}	7	{4,8}
3	{4,6}	8	{5,9}
4	{0,3,7}	9	{8}

COMPARING REPRESENTATIONS

- Adjacency list typically requires less space
- Is j a neighbour of i ?
  - → Check if $A[i,j] = 1$ in adjacency matrix
  - → Scan all neighbours of i in adjacency list
- Which are neighbours of i ?
  - → Scan all n entries in row i in adjacency matrix
  - → Takes time proportional to (out) degree of i in adj. list
- Choose representation depending on requirement

SUMMARY

- To operate on graphs, we need to represent them
- Adjacency matrix
  - → $n \times n$ matrix, $A[i,j] = 1$ iff $(i,j) \in E$
- Adjacency list
  - → For each vertex i, list of neighbours of i
- Can systematically explore a graph using these representation
  - → For reachability, propagate marking to all reachable vertices.

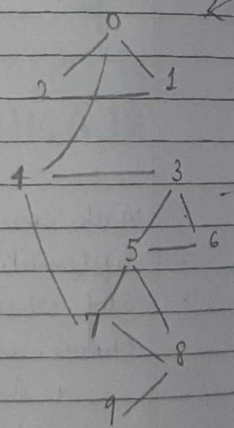
Breadth First Search

REACHABILITY IN A GRAPH

- Mark source vertex as reachable
- systematically mark neighbours of marked vertices
- Stop when target becomes marked
- Choose an appropriate representation
  - → Adjacency matrix
  - → Adjacency list
- Strategies for systematic exploration
  - → BFS : propagate marks in layers
  - → DFS : explore a path till it dies out, then backtrack

BREADTH FIRST SEARCH (BFS)

- Explore the graph level by level
  - → first visit vertices one step away.
  - → then two steps away
  - → ....
- Each visited vertex has to be explored
  - → Extend the search to its neighbours
  - → Do this only once for each vertex!
- Maintain information about vertices
  - → which vertices have been visited already
  - → among these, which are yet to be explored
- Assume $V = \{0, 1, \dots, n-1\}$



- visited : $V \rightarrow \{\text{True}, \text{False}\}$ tells us whether $v \in V$ has been visited
  - → initially, visited(v) = False for all $v \in V$
- Maintain a sequence of visited vertices yet to be explored
  - → a queue - first in, first out
  - → initially empty
- Exploring a vertex i
  - → For each edge (i, j), if visited (j) is False,
    - · set visited (j) to True
    - · append j to the queue
- Initially
  - → visited (v) = False for all $v \in V$
  - → queue of vertices to be explored is empty
- Start BFS from vertex j
  - → set visited(j) = True
  - → Add j to the queue
- Remove and explore vertex i at head of queue
  - → For each edge (i, j), if visited (j) is False
    - · set visited (j) to True
    - · Append j to the queue
- Stop when queue is empty

EXAMPLE : BFS from Vertex 7

	Visited
0	True
1	True
2	True
3	True
4	True
5	True
6	True
7	True
8	True
9	True

- Mark 7 & add to queue
- Explore 7, visit {4, 5, 8}
- Explore 4, visit {0, 3}
- Explore 5, visit {6}
- Explore 8, visit {9}
- Explore 0, visit {1, 2}
- Explore 3
- Explore 6
- Explore 9
- Explore 1
- Explore 2

To Explore Queue

7				
4	5	8		
5	8	0	3	
8	0	3	6	
0	3	6	9	
3	6	9	1	2
6	9	1	2	
9	1	2		
1	2			
2				

ENHANCING BFS TO RECORD PATHS

- If BFS from i sets visited(j) = True, we know that j is reachable from i
- How do we recover a path from i to j?
- visited(j) was set to True when exploring some vertex k
- Record parent(j) = k
- From j, follow parent links to trace back a path to i

Example: BFS from Vertex 7 with parent information

	Visited	Parent
0	True	4
1	True	0
2	True	0
3	True	4
4	True	7
5	True	7
6	True	5
7	True	
8	True	7
9	True	8

- Mark 7, add to queue
- Explore 7, visit {4,5,8}
- Explore 4, visit {0,3}
- Explore 5, visit {6}
- Explore 8, visit {9}
- Explore 0, visit {1,2}
- Explore 3
- Explore 6
- Explore 9
- Explore 1
- Explore 2

To explore Queue				
7				
4	5	8		
5	8	0	3	
8	0	3	6	
0	3	6	9	
3	6	9	1	2
6	9	1	2	
9	1	2		
1	2			
2				

Path from 7 to 6 is 7-5-6
Path from 7 to 2 is 7-4-0-2

ENHANCING BFS to RECORD DISTANCE

- BFS explores neighbours level by level
- By recording the level at which a vertex is visited, we get its distance from the source vertex
- Instead of visited(j), maintain level(j)
- Initialize level(j) = -1 for all j
- Set level(i) = 0 for source vertex
- If we visit j from k, set level(j) to level(k) + 1
- level(j) is the length of the shortest path from the source vertex, in no. of edges

EXAMPLE: BFS from vertex 7 with parent and distance information

	Level	Parent
0	2	4
1	3	0
2	3	0
3	2	4
4	1	7
5	1	7
6	2	5
7	0	
8	1	7
9	2	8

- Mark 7, add to queue
- Explore 7, visit {4,5,8}
- Explore 4, visit {0,3}
- Explore 5, visit {6}
- Explore 8, visit {9}
- Explore 0, visit {1,2}
- Explore 3
- Explore 6
- Explore 9
- Explore 1
- Explore 2

To Explore Queue				
7				
4	5	8		
5	8	0	3	
8	0	3	6	
0	3	6	9	
3	6	9	1	2
6	9	1	2	
9	1	2		
1	2			
2				

SUMMARY

- Breadth first search is a systematic strategy to explore a graph, level by level.
- Record which vertices have been visited
- Maintain visited but unexplored vertices in a queue
- Add parent info. to recover the path to each reachable vertex.
- Maintain level of information to record length of the shortest path, in terms of no. of edges.
- In general, edges are labelled with a cost (dist, time, ticket price, ...)
- Will look at weighted graphs, where shortest paths are in terms of cost, not no. of edges

February 14, 2021

Depth First Search

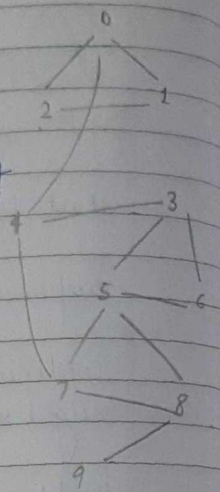
- Start from i, visit an ~~a~~ unexplored neighbour j
- suspend the exploration of i and explore j instead
- continue till you reach a vertex with no unexplored neighbours
- Backtrack to nearest suspended vertex that still has an unexplored neighbour
- Suspend vertices are stored in a stack
  - → Last in, first out
  - → Most recently suspended is checked first

<u>EXAMPLE</u> : DFS from vertex 4

	Visited	Stack of suspended vertices				
0	True	4				
1	True	4	0			
2	True	4	0	1		
3	True	4	0			
4	True	4				
5	True	4	3			
6	True	4	3	5		
7	True	4	3	5		
8	True	4	3	5	7	
9	True	4	3	5	7	8
		4	3	5	7	
		4	3	5		
		4	3			

- Mark 4, suspend 4, explore 0
- suspend 0, explore 1
- suspend 1, explore 2
- Backtrack to 1, 0, 4
- Suspend 4, explore 3
- Suspend 3, explore 5
- Suspend 5, explore 6
- Backtrack to 5,
- Suspend 5, explore 7
- Suspend 7, explore 8
- Suspend 8, explore 9
- Back track to 8, 7, 5, 3, 4

- Paths discovered by BFS are not shortest paths, unlike BFS
- Useful features can be found by recording the order in which DFS visits vertices
- DFS numbering - maintain a counter
  - → increment and record counter value each time you start and finish exploring a vertex
- DFS numbering can be used to
  - → Find cut vertices (deleting vertex disconnects graph)
  - → Find bridges (deleting edge disconnects graph)



SUMMARY

- DFS is another systematic strategy to explore a graph
- DFS uses a stack to suspend exploration and move to unexplored neighbours
- DFS numbering can be used to discover many facts about graphs

---

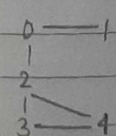
Applications of BFS & DFS

- BFS and DFS systematically compute reachability in graphs
- BFS works level by level
  - → Discovers shortest paths in terms of no. of edges
- DFS explores a vertex as soon as it is visited neighbours
  - → Suspend a vertex while exploring its neighbours
  - → DFS numbering describes the order in which vertices are explored
- Beyond reachability, what can we find out about a graph using BFS/DFS?

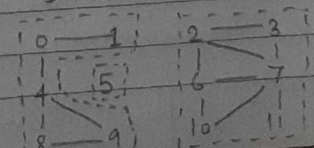
CONNECTIVITY

- A undirected graph is connected if every vertex is reachable from every other vertex.
- In a disconnected graph, we can identify the connected comp-onents
  - → Maximal subsets of vertices that are connected
  - → Isolated vertices are trivial components

CONNECTED GRAPH



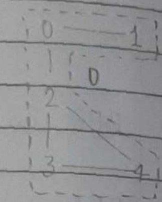
DISCONNECTED GRAPH



IDENTIFYING CONNECTED COMPONENTS

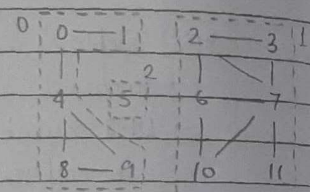
- Assign each vertex a component no.

Connected Graph



- Start BFS/DFS from vertex 0
  - → Initialise component no. to zero
  - → All visited nodes form a connected component
  - → Assign each visited node component number 0.

Disconnected Graph

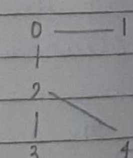


- Pick smallest unvisited node j
  - → increment component no. to 1
  - → Run BFS/DFS from node j
  - → Assign each visited node component no. 1

- Repeat until all nodes are visited.

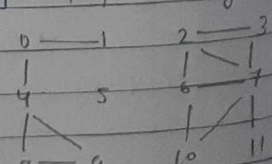
DETECTING CYCLES

Acyclic Graph



- A cycle is a path (technically, a walk) that starts & ends at the same vertex
  - → 4-8-9-4 is a cycle
  - → Cycle may repeat a vertex:
    2-3-7-10-6-7-2
  - → cycles should not repeat edges: i-j-i is not a cycle, i.e, 2-4-2
  - → simple cycle – only repeated vertices are start & end.

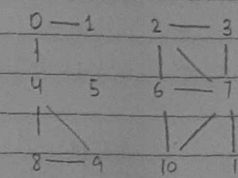
Graph with cycles



- A graph is acyclic if it has no cycles.

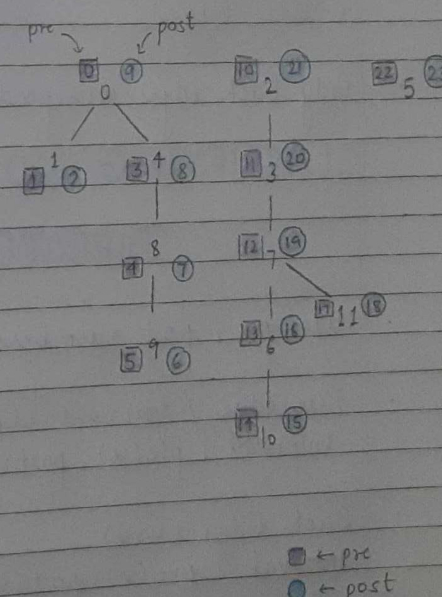
BFS tree

- A tree is a minimally connected graph
- Edges explored by BFS form a tree
  - → Technically, one tree per component
  - → Collection of trees is a forest



- Facts about trees
  - → A tree on n vertices has n-1 edges
  - → A tree is acyclic
- Any non tree edge creates a cycle
  - → detect cycles by searching for non tree edges

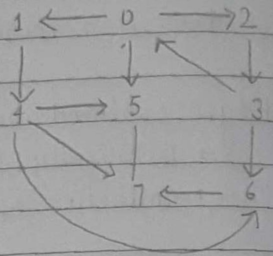
DFS tree



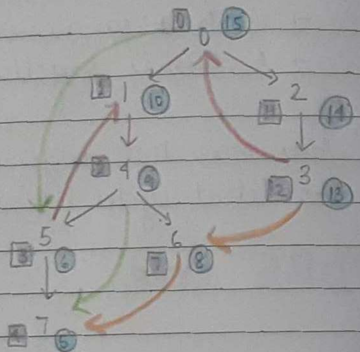
- Maintain a DFS counter, initially 0
- Increment counter each time we start & finish exploring a node
- Each vertex is assigned an entry number (pre) and exit no. (post)
- As before non-tree edges generates cycles

DIRECTED CYCLES

- In a directed graph, a cycle must follow same direction
  → 0 → 2 → ~~0~~3 → 0 is a cycle
  → 0 → 5 → 1 ← 0 is not



- Tree Edges
- Different type of non tree edges
  → Forward Edges
  → Back Edges
  → Cross Edges



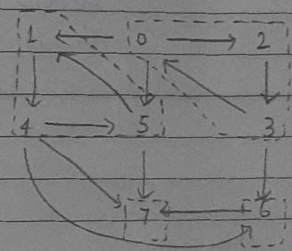
- Only back edges correspond to cycles

CLASSIFYING NON-TREE EDGES IN DIRECTED GRAPHS

- Use pre / post numbers
- Tree edge / forward edge (u,v)
  Interval [pre(u), post(u)] contains [pre(v), post(v)]
- Back Edge (u,v)
  Interval [pre(v), post(v)] contains [pre(u), post(u)]
- Cross Edge (u,v)
  Interval [pre(u), post(u)] and [pre(v), post(v)] are disjoint.

CONNECTIVITY IN DIRECTED GRAPHS

- Take directions into account
- Vertices i and j are strongly connected if there is a path from i to j and path from j to i
- Directed graphs can be decomposed into strongly connected components (SCC's)
  → within an SCC, each pair of vertices is strongly connected



- DFS numbering can be used to compute SCC's.

SUMMARY

- BFS and DFS can be used to identify connected components in an undirected graph
  → BFS and DFS identify an underlying tree, non-tree edges generate cycles.
- In a directed graph, non-tree edges can be forward / back / cross
  → only back edges generate cycles
  → classify non-tree edges using DFS numbering

- Directed graphs decompose into strongly connected components
  - → DFS numbering can be used to compute SCC decomposition
- DFS numbering can also be used to identify other features such as articulation points (cut vertices) and bridges (cut edges)
- Directed acyclic graphs are useful for representing dependencies
  - → Given course prerequisites, find a valid sequence to complete a programme.